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Properties of bounded representations for *G*-frames

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Abstract
The purpose of the paper is to analyze g-frames of the form $\{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$, where $T \in B(\mathcal{H})$ and $\varphi \in B(\mathcal{H}, \mathcal{K})$, and discuss the properties of the operator T. We consider stability of g-Riesz sequences of the form $\{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$. Finally, a weighted representation of a g frame is introduced and some of its properties are presented. We provide a sufficient condition for a given g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ to be represented by an operator $T \in B(\mathcal{H})$ and a sequence $\{a_i\}_{i=1}^{\infty}$.

Keywords Representation of a frame · g-Frame · Stability

Mathematics Subject Classification Primary 41A58 · 42C15 · 47A05

1 Introduction

Duffin and Schaeffer introduced frames in separable Hilbert spaces as an extension of orthonormal bases [15]. A frame does not necessarily contain linear independent vectors. Frames can be viewed as redundant bases which are generalization of orthonormal bases. They provide non-unique representations of vectors in a Hilbert space. The redundancy and flexibility of frames have led to their applications in various fields throughout mathematics and engineering, such as signal and image processing [4,5,16], data compression, dynamical sampling [1,2] and etc.

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Definition 1.1 A sequence $F = \{f_i\}_{i=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist two constants $A_F, B_F > 0$ such that

$$A_F\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B_F\|f\|^2, \quad f \in \mathcal{H}.$$

For background material on frame theory and related topics, we refer readers to [6,9, 14,17,19].

As introduced in[1,2] by Aldroubi et al. dynamic sampling deals with the frame properties of sequences $\{T^i f\}_{i=0}^{\infty}$, where $T : \mathcal{H} \to \mathcal{H}$ belongs to certain classes of linear operators and $f \in \mathcal{H}$.

Frames $\{f_i\}_{i=1}^{\infty}$ for which a representation of the form $\{T^i f\}_{i=0}^{\infty}$ with a bounded operator T, were characterized in [11]. While all linearly independent frames have a representation $\{T^i f\}_{i=0}^{\infty}$, it is more restrictive to obtain boundedness of the representing operator T. Christensen et al. [10,13] have shown that the only frames with bounded representations are those that are linearly independent and the kernel of their synthesis operators is invariant under right-shift operator $\mathcal{T} : \ell^2 \to \ell^2$ defined by

$$\mathcal{T}\big(\{c_i\}_{i=1}^{\infty}\big) = (0, c_1, c_2, \ldots).$$

For example, any Riesz sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ has the form $\{T^n f_n\}_{n=0}^{\infty}$ for some operator $T \in B(\mathcal{H})$ with closed range.

To study frames of the form $\{a_i T^i f_1\}_{i=0}^{\infty}$, they consider the weighted right-shift operator on $\ell^2(\mathcal{H}) := \Big\{\{c_i\}_{i=1}^{\infty} \subseteq \mathcal{H} : \ \sum_{i=1}^{\infty} \|c_i\|^2 < \infty\Big\}$ defined by

$$\mathcal{T}_\omega\big(\{c_i\}_{i=1}^{\infty}\big) = \Big(0, \frac{a_1}{a_2}c_1, \frac{a_2}{a_3}c_2, \ldots\Big),$$

for a sequence of non-zero scalars $\omega = \{a_i\}_{i=1}^{\infty}$ [12]. They have also explored the relationship between the representations of a frame and its duals. For the applications of frames, they established that frame representations were preserved under some perturbations. Results [2, Theorem 7] and [11, Proposition 3.5] are shown that the sequence $\{T^i f_1\}_{i=0}^{\infty}$ is not a frame, whenever T is unitary or compact. Also, Lemma 2.1 and Proposition 2.3 of [27] indicate the range of T is close and give some equivalent conditions for T to be surjective.

Sun [28] introduced a generalization of frames, named g-frames which are including some extensions and types of frames such as frames of subspaces [8], fusion frames [7], oblique frames [3], a class of time-frequency localization operators and generalized translation invariant (GTI) [20,21]. Therefore, some concepts presented in frames such as duality, stability and Riesz basis were also studied in g-frames [24,28,29].

Throughout this paper, J is countable set, $\mathbb{N}$ is natural numbers and $\mathbb{C}$ is complex numbers, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, $Id_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$, $B(\mathcal{H})$ and $GL(\mathcal{H})$ denote the set of bounded linear operators and invertible bounded linear operators on $\mathcal{H}$, respectively. Also, we apply $B(\mathcal{H}, \mathcal{K})$ for the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and consider $\varphi, \psi \in B(\mathcal{H}, \mathcal{K})$. We use $\ker T$

and $\mathcal{R}(T)$ for the null space and range $T \in B(\mathcal{H})$, respectively. Now, we summarize some facts about g-frames from [25,28]. For more on related subjects to g-frames, we refer to [18,23,26].

Definition 1.2 We say that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^{\infty}$ is a generalized frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i=1}^{\infty}$, or simply g-frame, if there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$A_\Lambda \|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 \leq B_\Lambda \|f\|^2, \;\; f \in \mathcal{H}. \tag{1.1}$$

We call A_Λ, B_Λ the lower and upper g-frame bounds, respectively. Λ is called a tight g-frame if $A_\Lambda = B_\Lambda$, and a Parseval g-frame if $A_\Lambda = B_\Lambda = 1$. A family Λ is called g-Bessel if the right hand inequality in (1.1) holds for all $f \in \mathcal{H}$, in this case, B_Λ is called the g-Bessel bound.

Example 1.3 [28] Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$. Suppose that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C})\}_{i=1}^{\infty}$, where

$$\Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H}.$$

It is easy to see that Λ is a g-frame.

For a g-frame Λ, there exists a unique positive and invertible operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ such that

$$S_\Lambda f = \sum_{i=1}^{\infty} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H},$$

and $A_\Lambda \cdot Id_{\mathcal{H}} \leq S_\Lambda \leq B_\Lambda \cdot Id_{\mathcal{H}}$. The operator S_Λ is called the g-frame operator for Λ. For a family $\{\mathcal{K}_i\}_{i=1}^{\infty}$ of Hilbert spaces, consider the space

$$\oplus_{i=1}^{\infty} \mathcal{K}_i = \left\{ \{g_i\}_{i=1}^{\infty} : g_i \in \mathcal{K}_i, \quad \sum_{i=1}^{\infty} \|g_i\|^2 < \infty \right\}.$$

For the case $\mathcal{K}_i = \mathcal{K}$ for all i, we use $\ell^2(\mathcal{K})$ instead of $\oplus_{i=1}^{\infty} \mathcal{K}_i$. It is clear that $\oplus_{i=1}^{\infty} \mathcal{K}_i$ is a Hilbert space with pointwise operations and with the inner product given by

$$\left\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \right\rangle = \sum_{i=1}^{\infty} \langle f_i, g_i \rangle.$$

For a g-Bessel Λ, the synthesis operator $T_\Lambda : \oplus_{i=1}^{\infty} \mathcal{K}_i \to \mathcal{H}$ is defined by

$$T_\Lambda \left(\{g_i\}_{i=1}^{\infty} \right) = \sum_{i=1}^{\infty} \Lambda_i^* g_i.$$

The analysis operator $T_\Lambda^*: \mathcal{H} \to \oplus_{i=1}^\infty \mathcal{K}_i$, adjoint of T_Λ, is given by

$$T_\Lambda^* f = \{\Lambda_i f\}_{i=1}^\infty, \quad f \in \mathcal{H}.$$

It is obvious that $S_\Lambda = T_\Lambda T_\Lambda^*$. For a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$, the sequence $\widetilde{\Lambda} = \{\widetilde{\Lambda} := \Lambda_i S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$, which is called canonical dual of Λ, is a g-frame with lower and upper g-frame bounds $\frac{1}{B_\Lambda}$ and $\frac{1}{A_\Lambda}$, respectively. For g-Bessel sequences Λ and Θ, we consider $S_{\Lambda\Theta} := T_\Lambda T_\Theta^*$.

Definition 1.4 Consider a sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$.

(i) We say that Λ is g-complete if $\cap_{i=1}^\infty \ker \Lambda_i = \{0\}$.
(ii) We say that Λ is a g-Riesz sequence if there are two constants $0 < A_\Lambda \le B_\Lambda < \infty$ such that for any finite sequence $\{g_i\}_{i=1}^n$,

$$A_\Lambda \sum_{i=1}^n \|g_i\|^2 \le \left\| \sum_{i=1}^n \Lambda_i^* g_i \right\|^2 \le B_\Lambda \sum_{i=1}^n \|g_i\|^2, \quad g_i \in \mathcal{K}_i.$$

(iii) We say that Λ is a g-Riesz basis if Λ is g-complete and g-Riesz sequence.
(iv) We say that Λ is a g-orthonormal basis if it satisfies the following:

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in \mathbb{N}, g_i \in \mathcal{K}_i, g_j \in \mathcal{K}_j,$$
$$\sum_{i=1}^\infty \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

A g-Riesz basis $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$ is g-biorthonormal with respect to its canonical dual $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$ in the following sense

$$\langle \Lambda_i^* g_i, \widetilde{\Lambda_j}^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in \mathbb{N}, g_i \in \mathcal{K}_i, g_j \in \mathcal{K}_j. \tag{1.2}$$

Theorem 1.5 [26] *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$ be a g-frame and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$ be a g-orthonormal basis. Then there is an onto bounded operator $V: \mathcal{H} \to \mathcal{H}$ such that $\Lambda_i = \Theta_i V^*$, for all $i \in \mathbb{N}$. If Λ is a g-Riesz basis, then V is invertible. If Λ is a g-orthonormal basis, then V is unitary.*

Theorem 1.6 [28] *Let for $i \in \mathbb{N}$, $\{e_{i,j}\}_{j \in J_i}$ be an orthonormal basis for $\mathcal{K}_i$. The sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i)\}_{i=1}^\infty$ is a g-frame (respectively, g-Bessel family, g-Riesz basis, g-orthonormal basis) if and only if $\{\Lambda_i^* e_{i,j}\}_{i \in \mathbb{N}, j \in J_i}$ is a frame (respectively, Bessel sequence, Riesz basis, orthonormal basis).*

Now we summarize some results of article [22] in which we generalize the results of articles [10,11] to introduce the representation of g-frames with bounded operators.

Definition 1.7 Let $T \in B(\mathcal{H})$. We say that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^\infty$ has a representation T if $\Lambda_i = \Lambda_1 T^{i-1}$ for all $i \in \mathbb{N}$. In the affirmative case, we say that Λ is represented by T.

Remark 1.8 Let $T \in B(\mathcal{H})$. Consider the frame $F = \{f_i\}_{i=1}^{\infty} = \{T^i f_1\}_{i=0}^{\infty}$ for $\mathcal{H}$, and the g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C})\}_{i=1}^{\infty}$, where $\Lambda_i f = \langle f, f_i \rangle$ for each $i \in \mathbb{N}$. It is clear that

$$\Lambda_{i+1} f = \langle f, f_{i+1} \rangle = \langle f, T f_i \rangle = \langle T^* f, f_i \rangle = \Lambda_i T^* f, \quad f \in \mathcal{H}.$$

Therefore, $\Lambda_i = \Lambda_1 (T^*)^{i-1}$ for all $i \in \mathbb{N}$, i.e., Λ is represented by T^*. Conversely, if $T \in B(\mathcal{H})$ and $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C})\}_{i=1}^{\infty} = \{\Lambda_1 T^i\}_{i=0}^{\infty}$ is a g-frame for $\mathcal{H}$, then by Riesz Representation Theorem there exists a sequence $\{f_i\}_{i=1}^{\infty}$ (which is a frame for $\mathcal{H}$) such that

$$f_i = (T^*)^{i-1} f_1, \quad \Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H}, \quad i \in \mathbb{N}.$$

The following theorem provides a sufficient condition for a given g-frame to be represented by an operator T. This theorem is a special case of Theorem 3.2.

Theorem 1.9 [22] *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a g-frame such that if $\sum_{i=1}^{n} \Lambda_i^* g_i = 0$ for some $n \in \mathbb{N}$, then $g_i = 0$ for every $1 \leq i \leq n$. Suppose that* $\ker T_\Lambda$ *is invariant under the right-shift operator. Then Λ is represented by $T \in B(\mathcal{H})$, where $\|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}$.*

Corollary 1.10 [22] *Every g-orthonormal basis and g-Riesz basis has a representation.*

Remark 1.11 [22] Consider a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ which is represented by T. For $S \in GL(\mathcal{H})$, the family $\Lambda S = \{\Lambda_i S \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ is a g-frame [26, Corollary 2.26], which is represented by $S^{-1} T S$.

In this paper, we generalize some recent results of [11,27] to obtain some properties of the operator $T \in B(\mathcal{H})$ in a g-frame $\{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$. We also generalize some results of [12,27] by introducing and investigating weighted representations for g-frames.

2 *G*-Frame representation properties

In this section, some properties of $T \in B(\mathcal{H})$ are provided when $\{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame.

Theorem 2.1 *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a g-frame represented by T. Then $\mathcal{R}(T^*) = \overline{\text{span}}\{T^* \Lambda_i^* e_j : j \in J\}_{i=1}^{\infty}$, where $\{e_j\}_{j \in J}$ is an orthonormal basis for K. In particular, $\mathcal{R}(T^*)$ and $\mathcal{R}(T)$ are closed.*

Proof By Theorem 1.6, $\{\Lambda_i^* e_j : j \in J\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, and so for every $f \in \mathcal{H}$, we have

$$T^* f = T^* \left(\sum_{i=1}^{\infty} \sum_{j \in J} c_{ij} \Lambda_i^* e_j \right) = \sum_{i=1}^{\infty} \sum_{j \in J} c_{ij} T^* \Lambda_i^* e_j.$$

Thus, $\mathcal{R}(T^*) \subseteq \overline{\text{span}}\{T^*\Lambda_i^* e_j : j \in J\}_{i=1}^{\infty} := \mathcal{H}_0$. On the other hand, since $\{T^*\Lambda_i^* e_j : j \in J\}_{i=1}^{\infty} = \{\Lambda_{i+1}^* e_j : j \in J\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}_0$, we have

$$g = \sum_{i=1}^{\infty} \sum_{j \in J} d_{ij} T^* \Lambda_i^* e_j = T^* \left(\sum_{i=1}^{\infty} \sum_{j \in J} d_{ij} \Lambda_i^* e_j \right), \quad g \in \mathcal{H}_0.$$

Then $\mathcal{R}(T^*) = \mathcal{H}_0$ is closed and so $\mathcal{R}(T)$ is closed. □

Proposition 2.2 *Let $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ be a g-frame such that $\|\varphi\| < \sqrt{A_\Lambda}$. Then T is injective.*

Proof We have

$$A_\Lambda \|f\|^2 \leq \sum_{i=0}^{\infty} \|\varphi T^i f\|^2 \leq \|\varphi\|^2 \left(\|f\|^2 + \sum_{i=1}^{\infty} \|T^i f\|^2 \right), \quad f \in \mathcal{H}.$$

Thus $\|\varphi\|^2 \sum_{i=1}^{\infty} \|T^i f\|^2 \geq \left(A_\Lambda - \|\varphi\|^2\right) \|f\|^2$ and since $\|\varphi\| < \sqrt{A_\Lambda}$, we infer that T is injective. □

The following example shows that, the other implication of Proposition 2.2 is not satisfied. Also, we give an example which is satisfied in Proposition 2.2 condition.

Example 2.3 (i) For $\varphi = 3Id_{\mathcal{H}}$ and $T = \frac{1}{2} Id_{\mathcal{H}}$, we get the tight g-frame $\Lambda = \{\varphi T^i \in B(\mathcal{H})\}_{i=0}^{\infty}$ with $\|\varphi\| > \sqrt{A_\Lambda}$.
(ii) For $\varphi = Id_{\mathcal{H}}$ and $T = \frac{1}{2} Id_{\mathcal{H}}$, we get the tight g-frame $\Lambda = \{\varphi T^i \in B(\mathcal{H})\}_{i=0}^{\infty}$ with $\|\varphi\| < \sqrt{A_\Lambda}$.

Theorem 2.4 *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a g-frame represented by T. Then the following are equivalent:*

(i) T *is injective.*
(ii) $\mathcal{R}(S_\Lambda^{-1}\Lambda_1^*) \cap \ker T = \{0\}$.
(iii) $\mathcal{R}(\Lambda_1^*) \subseteq \mathcal{R}(T^*)$.

Proof (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are clear. (ii) $\Rightarrow$ (i) Suppose that T is not injective. Then there exists $0 \neq f \in \ker T$, and we get

$$f = \sum_{i=1}^{\infty} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = S_\Lambda^{-1} \Lambda_1^* \Lambda_1 f + \sum_{i=1}^{\infty} S_\Lambda^{-1} \Lambda_{i+1}^* \Lambda_i T f = S_\Lambda^{-1} \Lambda_1^* \Lambda_1 f.$$

So $f \in \mathcal{R}(S_\Lambda^{-1}\Lambda_1^*)$, which is a contradiction. (iii) $\Rightarrow$ (i) For any $f \in \mathcal{H}$, we have

$$f = \sum_{i=1}^{\infty} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f = \Lambda_1^* \Lambda_1 S_\Lambda^{-1} f + \sum_{i=1}^{\infty} T^* \Lambda_i^* \Lambda_{i+1} S_\Lambda^{-1} f$$

$$= \Lambda_1^* \Lambda_1 S_\Lambda^{-1} f + T^* \left(\sum_{i=1}^{\infty} \Lambda_i^* \Lambda_{i+1} S_\Lambda^{-1} f \right).$$

Since $\mathcal{R}(\Lambda_1^*) \subseteq \mathcal{R}(T^*)$, we get $f \in \mathcal{R}(T^*)$. Therefore T^* is surjective, and so T is injective. □

The following example shows that the operator representation of a g-frame may not be injective.

Example 2.5 (i) Let $T \in B(\mathcal{H})$ and $f \in \mathcal{H}$ such that $F = \{T^i f\}_{i=0}^{\infty}$ be a Riesz basis for $\mathcal{H}$. Then by [27, Corollary 2.4], T is not surjective. If we consider $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C})\}_{i=1}^{\infty}$, where $\Lambda_i f = \langle f, T^i f \rangle$, then by Remark 1.8, Λ is represented by T^* which is not injective. On the other hand, since $\Lambda_i^*(1) = T^i f$ for any $i \in \mathbb{N}$, by Theorem 1.6, Λ is a g-Riesz basis.

(ii) Let $\mathcal{H}_0$ be a separable real Hilbert space and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis. Suppose $\Lambda = \{\Lambda_i \in B(\mathcal{H}_0, \mathbb{R})\}_{i=1}^{\infty}$ such that $\Lambda_i f := \langle f, e_i \rangle$ for all $f \in \mathcal{H}_0$. It is easy to see that Λ is a g-orthonormal basis. We have

(a) $\sum_{i=1}^{\infty} \|\Lambda_i f\|^2 = \sum_{i=1}^{\infty} |\langle e_i, f \rangle|^2 = \|f\|^2, \quad f \in \mathcal{H}_0$;
(b) $\Lambda_i^* \alpha = \alpha e_i$ for all $\alpha \in \mathbb{R}$ and

$$\left\| \sum_{i=1}^{n} \Lambda_i^* \alpha_i \right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2, \quad \{\alpha_i\}_{i=1}^{n} \subseteq \mathbb{R};$$

(c) $\langle \Lambda_i^* \alpha, \Lambda_j^* \beta \rangle = \delta_{i,j} \alpha\beta, \quad \alpha, \beta \in \mathbb{R}$.

If we define $V : \mathcal{H}_0 \mapsto \mathcal{H}_0$ by $Vf := \sum_{i=1}^{\infty} \langle f, e_{i+1} \rangle e_i$, then V is not injective and $\Lambda_{i+1} = \Lambda_i V$. Then Λ is represented by V.

Theorem 1.9 shows that for a g-frame $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ we have $\|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}$. The following theorem gives a sufficient condition to get $\|T\| \geq 1$. However, there is a g-frame $\{\varphi T^i \in B(\mathcal{H})\}_{i=0}^{\infty}$ with $\|T\| < 1$ (see Example 2.3).

Theorem 2.6 *Let $T \in B(\mathcal{H})$ and $\varphi \in B(\mathcal{H}, \mathcal{K})$ such that $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ be a g-frame. If $\bigcap_{i=0}^{n} \ker \varphi T^i \neq \{0\}$ for each $n \in \mathbb{N}$, then $\|T\| \geq 1$.*

Proof Let $\varepsilon > 0$, and suppose by contradiction that $\|T\| < 1$. Then there exists $N > 0$ such that $\sum_{i=N+1}^{\infty} \|\varphi T^i\|^2 < \varepsilon$. Let $0 \neq f \in \bigcap_{i=0}^{N} \ker \varphi T^i$ with $\|f\| = 1$. Then

$$A_\Lambda \leq \sum_{i=0}^{N} \|\varphi T^i f\|^2 + \sum_{i=N+1}^{\infty} \|\varphi T^i f\|^2 < \varepsilon.$$

Therefore $A_\Lambda = 0$, which is a contradiction. □

The main purpose of the reminder of this section is to show that the operator representation of g-frames may be compact but can not be unitary.

Theorem 2.7 *Let $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ be a g-frame. Then $T^n f \to 0$ as $n \to \infty$ for every $f \in \mathcal{H}$.*

Proof For every $n \in \mathbb{N}$ and $f \in \mathcal{H}$, we have

$$A_\Lambda \|T^n f\|^2 \leq \sum_{i=1}^{\infty} \|\varphi T^{i-1+n} f\|^2 = \sum_{i=n}^{\infty} \|\varphi T^i f\|^2. \tag{2.1}$$

Since $\sum_{i=0}^{\infty} \|\varphi T^i f\|^2$ is convergent, we get $\sum_{i=n}^{\infty} \|\varphi T^i f\|^2 \to 0$ as $n \to \infty$. Therefore, the inequality (2.1) implies that $T^n f \to 0$ as $n \to \infty$. □

Corollary 2.8 *For every unitary operator T and every $\varphi \in B(\mathcal{H}, \mathcal{K})$, the sequence $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ can not be a g-frame.*

Proof For every $f \in \mathcal{H}$,

$$\|f\| = \|(T^*)^n T^n f\| \leq \|T^*\|^n \|T^n f\| = \|T^n f\|. \tag{2.2}$$

If Λ is a g-frame, then by Theorem 2.7, $T^n f \to 0$ as $n \to \infty$, and so by the inequality (2.2), we get $f = 0$ which is a contradiction. □

Example 2.3 shows that the representation of a g-frame can be normal operator.

Corollary 2.9 *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be two g-orthonormal bases. Then for every $\varphi \in B(\mathcal{H}, \mathcal{K})$, the sequence $\Gamma = \{\varphi S_{\Lambda\Theta}^{i-1} \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ is not a g-frame.*

Proof By Theorem 1.5, there exists a unitary operator $U \in B(\mathcal{H})$ such that $\Theta_i = \Lambda_i U$. Then $T_\Theta = U^* T_\Lambda$ and

$$S_{\Lambda\Theta} S_{\Lambda\Theta}^* = T_\Lambda T_\Theta^* T_\Theta T_\Lambda^* = T_\Lambda T_\Lambda^* U U^* T_\Lambda T_\Lambda^* = S_\Lambda Id_{\mathcal{H}} S_\Lambda = Id_{\mathcal{H}}.$$

Similary, we get $S_{\Lambda\Theta}^* S_{\Lambda\Theta} = Id_{\mathcal{H}}$. So $S_{\Lambda\Theta}$ is a unitary operator on $\mathcal{H}$ and by Corollary 2.8, Γ is not a g-frame for every $\varphi \in B(\mathcal{H}, \mathcal{K})$. □

Proposition 2.10 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Assume that $T \in B(\mathcal{H}_1)$, $S \in B(\mathcal{H}_2)$, $\varphi \in B(\mathcal{H}_1, \mathcal{K})$ and $\psi \in B(\mathcal{H}_2, \mathcal{K})$ such that $T = V^{-1} S V$ and $\psi V = \varphi$ for some $V \in GL(\mathcal{H}_1, \mathcal{H}_2)$. Then $\{\varphi T^i \in B(\mathcal{H}_1, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame if and only if $\{\psi S^i \in B(\mathcal{H}_2, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame. In the affirmative case V is unique.*

Proof For every $f \in \mathcal{H}_1$, we have

$$\begin{aligned}\sum_{i=0}^{\infty} \|\varphi T^i f\|^2 = \sum_{i=0}^{\infty} \|\psi V (V^{-1} S V)^i f\|^2 &= \sum_{i=0}^{\infty} \|\psi V V^{-1} S^i V f\|^2 \\ &= \sum_{i=0}^{\infty} \|\psi S^i V f\|^2.\end{aligned}$$

Since $V \in GL(\mathcal{H}_1, \mathcal{H}_2)$, the sequence $\{\varphi T^i \in B(\mathcal{H}_1, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame if and only if $\Lambda = \{\psi S^i \in B(\mathcal{H}_2, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame. Moreover, if $\{\varphi T^i \in B(\mathcal{H}_1, \mathcal{K})\}_{i=0}^{\infty}$ is a g-frame and $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{K}$, then by Theorem 1.6, each $f \in \mathcal{H}_1$ can be represented by $f = \sum_{i=0}^{\infty}\sum_{j\in J} c_{ij}(T^i)^*\varphi^* e_j$ for some $\{c_{ij} : j \in J\}_{i=0}^{\infty} \in \ell^2$. Hence

$$\begin{aligned}(V^*)^{-1} f &= (V^*)^{-1}\left(\sum_{i=0}^{\infty}\sum_{j\in J} c_{ij}(T^i)^*\varphi^* e_j\right)\\ &= (V^*)^{-1}\left(\sum_{i=0}^{\infty}\sum_{j\in J} c_{ij}V^*(S^{i-1})^*(V^{-1})^*V^*\psi^* e_j\right)\\ &= \sum_{i=0}^{\infty}\sum_{j\in J} c_{ij}(S^i)^*\psi^* e_j.\end{aligned}$$

Therefore V is unique. □

Proposition 2.11 *Let* $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ *be a g-frame. If the sequence* $\{\varphi S_\Lambda^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ *is a g-frame for some* $\varphi \in B(\mathcal{H}, \mathcal{K})$*, then* $A_\Lambda < 1$.

Proof The proof is the same as the proof of the [27, Proposition 2.7]. □

In [27, Corollary 2.4], it has been shown that for Riesz basis $\{T^i f_1\}_{i=0}^{\infty}$ the operator T can not be surjective. A result in [11, Proposition 3.5] and [27, Proposition 2.2] states that if $\{T^i f\}_{i=0}^{\infty}$ is a frame for an infinite dimensional $\mathcal{H}$ with $T \in B(\mathcal{H})$, then T can not be compact. The following proposition provides a generalization of this result.

Proposition 2.12 *Let* $\dim \mathcal{K} < \infty$ *and* $\dim \mathcal{H} = \infty$. *If* $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ *is a g-frame represented by* T*, then* T *is not compact.*

Proof Let $\{e_j\}_{j=1}^{m}$ be an orthonormal basis for $\mathcal{K}$ and T be compact. By Theorem 2.1, $\mathcal{R}(T^*) = \overline{\text{span}}\{\Lambda_{i+1}^* e_j : 1 \le j \le m\}_{i=1}^{\infty}$, and therefore by [9, Lemma 2.5.1], there exists $T^\dagger \in B(\mathcal{H})$ such that $T^*T^\dagger = Id_{\mathcal{R}(T^*)}$. Since T is compact, T^* is compact and so $\mathcal{R}(T^*)$ is finite-dimensional. Consequently, by Theorem 1.6 we get $\mathcal{H} = \overline{\text{span}}\{\Lambda_i^* e_j : 1 \le j \le m\}_{i=1}^{\infty}$. This implies that $\mathcal{H}$ is finite-dimensional which is a contradiction. □

The following example shows that the assumption $\dim \mathcal{K} < \infty$ in Proposition 2.12 is necessary.

Example 2.13 Let $T : \ell^2 \to \ell^2$ be an operator defined by $T\{a_n\}_{n=1}^{\infty} = (\alpha a_1, 0, 0, ...)$ which $|\alpha| < 1$ is a fixed scalar. It is clear that T is compact and $\Lambda = \{T^i \in B(\ell^2)\}_{i=0}^{\infty}$ is a g-frame. In fact, for every $\{a_n\}_{n=1}^{\infty} \in \ell^2$, we have

$$\sum_{i=0}^{\infty} \|T^i\{a_n\}_{n=1}^{\infty}\|^2 = \|\{a_n\}_{n=1}^{\infty}\|^2 + \sum_{i=1}^{\infty} \|T^i\{a_n\}_{n=1}^{\infty}\|^2$$

$$= \|\{a_i\}_{n=1}^{\infty}\|^2 + \sum_{i=1}^{\infty} \|(\alpha^i a_1, 0, 0, \ldots)\|^2.$$

Then $\|\{a_n\}_{n=1}^{\infty}\|^2 \leq \sum_{i=0}^{\infty} \|T^i\{a_n\}_{n=1}^{\infty}\|^2 \leq \dfrac{1}{1-\alpha^2}\|\{a_n\}_{n=1}^{\infty}\|^2$.

In the following theorem, by applying a perturbation, we consider a sequence of operators Θ in some sense close to the g-Riesz sequence Λ, and then get some conditions for Θ to be g-Riesz sequence.

Theorem 2.14 *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a g-Riesz sequence and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a sequence of operators. Suppose that S_Λ is the g-frame operator of Λ (as a g-frame sequence) such that $\alpha := \sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i\|\|\Lambda_1 S_\Lambda^{-1}\| < 1$ and $\beta := \sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i\|^2 < \infty$. Then Θ is a g-Riesz sequence.*

Proof For every $\{g_i\}_{i=1}^{\infty} \in \ell^2(\mathcal{K})$, we have

$$\begin{aligned}
\left\|\sum_{i=1}^{\infty} \Theta_i^* g_i\right\| &= \left\|\sum_{i=1}^{\infty}(\Theta_i^* - \Lambda_i^*)g_i + \sum_{i=1}^{\infty} \Lambda_i^* g_i\right\| \\
&\leq \sum_{i=1}^{\infty} \|\Theta_i^* - \Lambda_i^*\|\|g_i\| + \left\|\sum_{i=1}^{\infty} \Lambda_i^* g_i\right\| \\
&\leq \left(\sum_{i=1}^{\infty} \|\Theta_i - \Lambda_i\|^2\right)^{\frac{1}{2}} \|\{g_i\}_{i=1}^{\infty}\| + \sqrt{B_\Lambda}\|\{g_i\}_{i=1}^{\infty}\| \\
&\leq \left(\sqrt{\beta} + \sqrt{B_\Lambda}\right) \|\{g_i\}_{i=1}^{\infty}\|. \qquad (2.3)
\end{aligned}$$

By the assumption, Λ is a g-frame for $\mathcal{M} = \overline{\text{span}}\{\Lambda_i^*(\mathcal{K})\}_{i=1}^{\infty}$. Let $U : \mathcal{H} \to \mathcal{H}$ be defined by

$$Uf = \sum_{i=1}^{\infty} \Theta_i^* \Lambda_i S_\Lambda^{-1} P_{\mathcal{M}} f, \quad f \in \mathcal{H},$$

where $P_{\mathcal{M}} : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection on $\mathcal{M}$. Since $\{\Lambda_i S_\Lambda^{-1} P_{\mathcal{M}} f\}_{i=1}^{\infty} \in \ell^2(\mathcal{K})$, by (2.3) we have

$$\|Uf\| \leq \left(\sqrt{\beta} + \sqrt{B_\Lambda}\right)\left\|\left\{\Lambda_i S_\Lambda^{-1} P_M f\right\}_{i=1}^{\infty}\right\| \leq \frac{\sqrt{\beta} + \sqrt{B_\Lambda}}{\sqrt{A_\Lambda}}\|f\|, \quad f \in \mathcal{H}.$$

Note that the operator U on $\mathcal{M}$ is equal to $S_{\Theta\Lambda} S_\Lambda^{-1}$. Let $f \in \mathcal{M}$, then

$$\|f - Uf\| = \left\|\sum_{i=1}^{\infty} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f - \sum_{i=1}^{\infty} \Theta_i^* \Lambda_i S_\Lambda^{-1} f\right\|$$

$$= \left\| \sum_{i=1}^{\infty} (\Lambda_i^* - \Theta_i^*) \Lambda_i S_\Lambda^{-1} f \right\|$$
$$\leq \sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i\| \|\Lambda_i S_\Lambda^{-1}\| \|f\| = \alpha \|f\|.$$

This implies that $\|Uf\| \geq (1-\alpha)\|f\|$ for all $f \in \mathcal{M}$. On the other hand by applying (1.2), we get $U\Lambda_k^* = \Theta_k^*$ for each $k \in \mathbb{N}$, because

$$\langle U\Lambda_k^* g, f\rangle = \sum_{i=1}^{\infty} \langle \Theta_i^* \Lambda_i S_\Lambda^{-1} \Lambda_k^* g, f\rangle = \sum_{i=1}^{\infty} \langle \Lambda_k^* g, S_\Lambda^{-1} \Lambda_i^* \Theta_i f\rangle$$
$$= \langle g, \Theta_k f\rangle = \langle \Theta_k^* g, f\rangle, \quad f \in \mathcal{H},\ g \in \mathcal{K}.$$

Consequently, for each $\{g_i\}_{i=1}^{\infty} \in \ell^2(\mathcal{K})$ we have

$$\left\| \sum_{i=1}^{\infty} \Theta_i^* g_i \right\| = \left\| \sum_{i=1}^{\infty} U\Lambda_i^* g_i \right\| = \left\| U \sum_{i=1}^{\infty} \Lambda_i^* g_i \right\|$$
$$\geq (1-\alpha) \left\| \sum_{i=1}^{\infty} \Lambda_i^* g_i \right\| \geq (1-\alpha)\sqrt{A_\Lambda} \left(\sum_{i=1}^{\infty} \|g_i\|^2 \right)^{1/2}.$$

□

Theorem 2.15 *Let $T \in B(\mathcal{H})$ and $\varphi, \psi \in B(\mathcal{H}, \mathcal{K})$. Suppose that $\Lambda = \{\varphi T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ be a g-Riesz sequence and there exists $\mu \in [0, 1)$ such that $\|\psi\| < (1-\mu)\sqrt{A_\Lambda}$ and $\|\psi T^i\| \leq \mu^i \|\psi\|$ for each $i \in \mathbb{N}$. Then $\{(\varphi + \psi)T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ is a g-Riesz sequence.*

Proof It is sufficient to show that the sequence $\{(\varphi + \psi)T^i \in B(\mathcal{H}, \mathcal{K})\}_{i=0}^{\infty}$ satisfies the conditions of Theorem 2.3. Let S_Λ be the g-frame operator of Λ (as a g-frame sequence). It is clear that $\|\varphi S_\Lambda^{-1}\| \leq \frac{1}{\sqrt{A_\Lambda}}$. By the assumptions, we get

$$\sum_{i=0}^{\infty} \|(\varphi + \psi)T^i - \varphi T^i\|^2 = \sum_{i=0}^{\infty} \|\psi T^i\|^2 \leq \sum_{i=0}^{\infty} \mu^{2i} \|\psi\|^2 = \frac{\|\psi\|^2}{1-\mu^2},$$
$$\sum_{i=0}^{\infty} \|\psi T^i\| \|\varphi S_\Lambda^{-1}\| \leq \frac{\|\psi\|}{(1-\mu)\sqrt{A_\Lambda}} < 1.$$

Therefore the proof is completed. □

3 *G*-Frame representation with a bounded operator and a sequence of non-zero scalars

Frames of the form $\{a_i T^i f\}_{i=0}^{\infty}$ for some non-zero scalars with $\sup_{i\in\mathbb{N}}\left|\frac{a_i}{a_{i+1}}\right| < \infty$ and $T \in B(\mathcal{H})$, were introduced and investigated in [12,27]. In this section, we introduce this kind of representation for g-frames.

Definition 3.1 We say that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ has a weighted representation if there are a sequence of non-zero scalars $\{a_i\}_{i=1}^{\infty}$ and $T \in B(\mathcal{H})$ such that $\Lambda_i = a_i \Lambda_1 T^{i-1}$ for all $i \in \mathbb{N}$. In the affirmative case, we say that Λ is represented by $\big(T, \{a_i\}_{i=1}^{\infty}\big)$.

Note that for $\{a_i\}_{i=1}^{\infty} = \{1\}_{i=1}^{\infty}$ in Definition 3.1, we have [22, Definition 2.2]. Also, It is obvious that if a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ is represented by $\big(T, \{a_i\}_{i=1}^{\infty}\big)$, then $a_1 = 1$, and

$$\Lambda_{i+1} = a_{i+1}\Lambda_1 T^i = \frac{a_{i+1}}{a_i} a_i \Lambda_1 T^{i-1} T = \frac{a_{i+1}}{a_i}\Lambda_i T, \quad i \in \mathbb{N}.$$

In [22], it is shown that the g-frame $\Lambda = \{\Lambda_n \in B(\mathbb{C})\}_{n=1}^{\infty}$ with $\Lambda_n = \frac{1}{n^4+1} Id_{\mathbb{C}}$ has not any representation, but this g-frame is represented by $\big(Id_{\mathbb{C}}, \{a_n\}_{n=1}^{\infty}\big)$, where $a_n = \frac{2}{n^4+1}$ for all $n \in \mathbb{N}$.

Theorem 3.2 *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ be a g-frame such that if $\sum_{i=1}^{n} \Lambda_i^* g_i = 0$ for some $n \in \mathbb{N}$, then $g_i = 0$ for every $1 \le i \le n$. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of non-zero scalars with $\mu := \sup_{i\in\mathbb{N}}\left|\frac{a_i}{a_{i+1}}\right| < \infty$, and* $\ker T_\Lambda$ *be invariant under the weighted right-shift operator $\mathcal{T}_\omega$, where $\omega = \{\overline{a_i}\}_{i=1}^{\infty}$. Then Λ is represented by $\big(T, \{a_i\}_{i=1}^{\infty}\big)$ where $\|T\| \le \mu\sqrt{B_\Lambda A_\Lambda^{-1}}$.*

Proof Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{K}$. We define the linear map S : $\operatorname{span}\{\Lambda_i^*(\mathcal{K})\}_{i=1}^{\infty} \to \operatorname{span}\{\Lambda_i^*(\mathcal{K})\}_{i=1}^{\infty}$ with

$$S\big(\Lambda_i^* e_j\big) = \frac{\overline{a_i}}{\overline{a_{i+1}}}\Lambda_{i+1}^* e_j.$$

By the assumption, S is well-defined. Now, we show that S is bounded. Let $f = \sum_{i\in I, j\in G} c_{ij}\Lambda_i^* e_j \in \operatorname{span}\{\Lambda_i^*(\mathcal{K})\}_{i=1}^{\infty}$ where $I \subseteq \mathbb{N}$ and $G \subseteq J$ are non-empty finite sets. We may assume that $\{c_{ij} : j \in J\}_{i=1}^{\infty} \in \ell^2(\mathbb{N}\times J)$ by letting $c_{ij} = 0$ if $(i, j) \notin I \times G$. By Theorem 1.6, $F = \{\Lambda_i^* e_j : j \in J\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A_Λ and B_Λ, respectively. We can write

$$\{c_{ij} : j \in J\}_{i=1}^{\infty} = \{d_{ij} : j \in J\}_{i=1}^{\infty} + \{r_{ij} : j \in J\}_{i=1}^{\infty}$$

where $\{d_{ij} : j \in J\}_{i=1}^{\infty} \in \ker T_F$ and $\{r_{ij} : j \in J\}_{i=1}^{\infty} \in (\ker T_F)^{\perp}$. Then $\left\{\sum_{j\in J} d_{ij}e_j\right\}_{i=1}^{\infty}$ belongs to $\ker T_\Lambda$, and by the assumption we conclude that

$$\sum_{i=1}^{\infty}\sum_{j\in J}\frac{\overline{a_i}}{a_{i+1}}d_{ij}\Lambda_{i+1}^{*}e_j = T_\Lambda \mathcal{T}_\omega\left(\left\{\sum_{j\in J} d_{ij}e_j\right\}_{i=1}^{\infty}\right) = 0.$$

Therefore

$$\begin{aligned}\|Sf\|^2 &= \left\|\sum_{i=1}^{\infty}\sum_{j\in J}\frac{\overline{a_i}}{a_{i+1}}c_{ij}\Lambda_{i+1}^{*}e_j\right\|^2 = \left\|\sum_{i=1}^{\infty}\sum_{j\in J}\frac{\overline{a_i}}{a_{i+1}}r_{ij}\Lambda_{i+1}^{*}e_j\right\|^2 \\ &\le \mu^2 B_\Lambda \sum_{i=1}^{\infty}\sum_{j\in J}|r_{ij}|^2. \end{aligned} \tag{3.1}$$

Since $\{r_{ij} : j \in J\}_{i=1}^{\infty} \in (\ker T_F)^{\perp}$, by [9, Lemma 5.5.5], we have

$$A_\Lambda \sum_{i=1}^{\infty}\sum_{j\in J}|r_{ij}|^2 \le \left\|\sum_{i=1}^{\infty}\sum_{j\in J} r_{ij}\Lambda_i^{*}e_j\right\|^2. \tag{3.2}$$

Hence by the inequalities (3.1) and (3.2), we have

$$\begin{aligned}\|Sf\|^2 &\le \mu^2 B_\Lambda A_\Lambda^{-1}\left\|\sum_{i=1}^{\infty}\sum_{j\in J} r_{ij}\Lambda_i^{*}e_j\right\|^2 \\ &= \mu^2 B_\Lambda A_\Lambda^{-1}\left\|\sum_{i=1}^{\infty}\sum_{j\in J} (d_{ij}+r_{ij})\Lambda_i^{*}e_j\right\|^2 \\ &= \mu^2 B_\Lambda A_\Lambda^{-1}\left\|\sum_{i=1}^{\infty}\sum_{j\in J} c_{ij}\Lambda_i^{*}e_j\right\|^2 = \mu^2 B_\Lambda A_\Lambda^{-1}\|f\|^2.\end{aligned}$$

So, S is bounded and can be extended to $Q \in B(\mathcal{H})$. Let $T = Q^*$, then it is obvious that Λ is represented by $\left(T, \{a_i\}_{i=1}^{\infty}\right)$ and $\|T\| \le \mu\sqrt{B_\Lambda A_\Lambda^{-1}}$. □

Theorem 3.3 *Let* $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K})\}_{i=1}^{\infty}$ *and* $\Theta = \{\Theta_i \in B(\mathcal{H},\mathcal{K})\}_{i=1}^{\infty}$ *be sequences such that*

$$f = \sum_{i=1}^{\infty}\Lambda_i^{*}\Theta_i f, \quad f \in \mathcal{H}. \tag{3.3}$$

Assume that $\{a_i\}_{i=1}^{\infty}$ *is a sequence of non-zero scalars such that for every* $f \in \mathcal{H}$ *the series* $\sum_{i=1}^{\infty} \frac{\overline{a_i}}{\overline{a_{i+1}}} \Lambda_{i+1}^* \Theta_i f$ *converges. Then* $\Lambda = \{a_i \Lambda_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ *for some* $T \in B(\mathcal{H})$ *if and only if*

$$\Lambda_{j+1} = \frac{a_{j+1}}{a_j} \sum_{i=1}^{\infty} \frac{a_i}{a_{i+1}} \Lambda_j \Theta_i^* \Lambda_{i+1}, \quad j \in \mathbb{N}. \tag{3.4}$$

Proof First, we assume that $\Lambda = \{a_i \Lambda_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K})\}_{i=1}^{\infty}$ for some $T \in B(\mathcal{H})$. Then $\Lambda_{i+1} = \frac{a_{i+1}}{a_i} \Lambda_i T$ for all $i \in \mathbb{N}$. By (3.3), we get

$$T^* f = \sum_{i=1}^{\infty} T^* \Lambda_i^* \Theta_i f = \sum_{i=1}^{\infty} (\Lambda_i T)^* \Theta_i f = \sum_{i=1}^{\infty} \frac{\overline{a_i}}{\overline{a_{i+1}}} \Lambda_{i+1}^* \Theta_i f, \quad f \in \mathcal{H}.$$

Then

$$\Lambda_{j+1}^* g = \frac{\overline{a_{j+1}}}{\overline{a_j}} T^* \Lambda_j^* g = \frac{\overline{a_{j+1}}}{\overline{a_j}} \sum_{i=1}^{\infty} \frac{\overline{a_i}}{\overline{a_{i+1}}} \Lambda_{i+1}^* \Theta_i \Lambda_j^* g, \quad g \in \mathcal{K}.$$

Therefore (3.4) is concluded. For the other implication, let (3.4) hold. We define the linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$T f = \sum_{i=1}^{\infty} \frac{a_i}{a_{i+1}} \Theta_i^* \Lambda_{i+1} f, \quad f \in \mathcal{H}.$$

By the assumption and uniform boundedness principle, T is well-defined and bounded. Then by (3.4), for every $f \in \mathcal{H}$, we have

$$\Lambda_j T f = \sum_{i=1}^{\infty} \frac{a_i}{a_{i+1}} \Lambda_j \Theta_i^* \Lambda_{i+1} f = \frac{a_j}{a_{j+1}} \Lambda_{j+1} f, \quad j \in \mathbb{N}.$$

This completes the proof. □

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