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G-frame representations with bounded operators

Yavar Khedmati* and Fatemeh Ghobadzadeh†

Department of Mathematics, Faculty of Sciences
University of Mohaghegh Ardabili
Ardabil 56199-11367, Iran
**khedmati.y@uma.ac.ir; khedmatiy.y@gmail.com*
†*gobadzadehf@yahoo.com*

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Dynamical sampling, as introduced by Aldroubi *et al.*, deals with frame properties of sequences of the form $\{T^{i-1}f_1\}_{i\in\mathbb{N}}$, where f_1 belongs to Hilbert space $\mathcal{H}$ and $T:\mathcal{H}\to\mathcal{H}$ belongs to certain classes of bounded operators. Christensen *et al.* studied frames for $\mathcal{H}$ with index set $\mathbb{N}$ (or $\mathbb{Z}$), that has representations in the form $\{T^{i-1}f_1\}_{i\in\mathbb{N}}$ (or $\{T^i f_0\}_{i\in\mathbb{Z}}$). As frames of subspaces, fusion frames and generalized translation invariant systems are the special cases of g-frames, the purpose of this paper is to study and get sufficient conditions for g-frames $\Lambda=\{\Lambda_i\in B(\mathcal{H},\mathcal{K}):i\in\mathbb{N}\text{ (or }\mathbb{Z})\}$ having the form $\Lambda_{i+1}=\Lambda_1T^i$, $T\in B(\mathcal{H})$ (or $\Lambda_{i+1}=\Lambda_0T^i$, $T\in GL(\mathcal{H})$). Also, we get the relation between representations of dual g-frames with index set $\mathbb{Z}$. Finally, we study stability of g-frame representations under some perturbations.

Keywords: Representation; g-frame; dual; stability.

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1. Introduction

In 1952, the concept of frames for Hilbert spaces was defined by Duffin and Schaeffer.[14] Frames are important tools in the signal/image processing,[3,4,15] data compression,[13,22] dynamical sampling,[1,2] etc.

Throughout this paper, I and J are countable sets, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, $\{\mathcal{K}_i : i\in I\}$ is a family of separable Hilbert spaces, $\mathrm{Id}_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$, $B(\mathcal{H})$ and $GL(\mathcal{H})$ denote the set of all bounded linear operators and the set of all invertible bounded linear operators on $\mathcal{H}$, respectively, and $l^2(\mathcal{H},I)=\{\{g_i\}_{i\in I}:g_i\in\mathcal{H},\sum_{i\in I}\|g_i\|^2<\infty\}$. Also, we will apply $B(\mathcal{H},\mathcal{K})$ for the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We use $\ker T$ and $\operatorname{ran} T$ for the null space and range of $T\in B(\mathcal{H})$, respectively. We denote the natural, integer and complex numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$, respectively.

A sequence $F = \{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist two constants $A_F, B_F > 0$ such that

$$A_F\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B_F\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

Let $F = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$, then the operator

$$T_F : l^2(\mathbb{C}, I) \to \mathcal{H}, \quad T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i,$$

is well-defined and onto, also its adjoint is

$$T_F^* : \mathcal{H} \to l^2(\mathbb{C}, I), \quad T_F^* f = \{\langle f, f_i\rangle\}_{i\in I}.$$

The operators T_F and T_F^* are called the synthesis and analysis operators of F, respectively.

Frames for $\mathcal{H}$ allow each $f \in \mathcal{H}$ to be expanded as an (infinite) linear combination of the frame elements. A frame $G = \{g_i\}_{i\in I}$ such that for every $f \in \mathcal{H}$ we have

$$\sum_{i\in I} \langle f, f_i\rangle g_i = f,$$

is called dual of frame $F = \{f_i\}_{i\in I}$. For more on frames, we refer to Refs. 7 and 17.

Aldroubi *et al.* introduced the concept of dynamical sampling which dealt with frame properties of sequences of the form $\{T^i f_1\}_{i\in\mathbb{N}}$, for $f_1 \in \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ belonging to certain classes of bounded operators.[1,2] Christensen and Hassannasab analyze frames $F = \{f_i\}_{i\in\mathbb{Z}}$ having the form $F = \{T^i f_0\}_{i\in\mathbb{Z}}$, where T is a bijective linear operator (not necessarily bounded) on $\mathrm{span}\{f_i\}_{i\in\mathbb{Z}}$. They show, $(T^*)^{-1}$ is the only possibility of the representing operator for the duals of the frame $F = \{f_i\}_{i\in\mathbb{Z}} = \{T^i f_0\}_{i\in\mathbb{Z}}$, $T \in GL(\mathcal{H})$.[9] They even clarify stability of the representation of frames. Christensen *et al.* determine the frames that have a representation with a bounded operator and survey the properties of this operator.[12]

Proposition 1.1 (Ref. 10). *Consider a frame sequence $F = \{f_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$ which spans an infinite-dimensional subspace. The following is equivalent:*

(i) *F is linearly independent.*
(ii) *There exists a linear operator $T : \mathrm{span}\{f_i\}_{i\in\mathbb{N}} \to \mathcal{H}$ such that $\{f_i\}_{i\in\mathbb{N}} = \{T^{i-1} f_1\}_{i\in\mathbb{N}}$.*

The right-shift operator on $l^2(\mathcal{H}, \mathbb{N})$ and $l^2(\mathcal{H}, \mathbb{Z})$, is defined by

$\mathcal{T}(\{c_i\}_{i\in\mathbb{N}}) = (0, c_1, c_2, \ldots)$ and $\mathcal{T}(\{c_i\}_{i\in\mathbb{Z}}) = \{c_{i-1}\}_{i\in\mathbb{Z}}$, respectively. Clearly, the right-shift operator on $l^2(\mathcal{H}, \mathbb{Z})$ is unitary and $\mathcal{T}^*$ is the left-shift operator, i.e. $\mathcal{T}^*(\{c_i\}_{i\in\mathbb{Z}}) = \{c_{i+1}\}_{i\in\mathbb{Z}}$. A subspace $V \subseteq l^2(\mathcal{H}, \mathbb{N})$ is invariant under the right-shift operator if $\mathcal{T}(V) \subseteq V$ and a subspace $V \subseteq l^2(\mathcal{H}, \mathbb{Z})$ is invariant under the right-shift (left-shift) operator if $\mathcal{T}(V) \subseteq V$ $(\mathcal{T}^*(V) \subseteq V)$.

Theorem 1.2 (Ref. 12). *Consider a frame $F = \{f_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$. Then the following is equivalent:*

(i) *F has a representation $F = \{T^{i-1} f_1\}_{i\in\mathbb{N}}$ for some $T \in B(\mathcal{H})$.*
(ii) *For some dual frame $G = \{g_i\}_{i\in\mathbb{N}}$ (and hence all)*

$$f_{j+1} = \sum_{i\in\mathbb{N}} \langle f_j, g_i\rangle f_{i+1}, \quad \forall\, j \in \mathbb{N}.$$

(iii) *The $\ker T_F$ is invariant under the right-shift operator.*

In the affirmative case, let $G = \{g_i\}_{i\in\mathbb{N}}$ denote an arbitrary dual frame of F, the operator T has the form

$$Tf = \sum_{i\in\mathbb{N}} \langle f, g_i\rangle f_{i+1}, \quad \forall\, f \in \mathcal{H},$$

and $1 \le \|T\| \le \sqrt{B_F A_F^{-1}}$.

In 2006, generalized frames (or simply g-frames) and g-Riesz bases were introduced by Sun.[23] "G-frames are natural generalizations of frames which cover many other recent generalizations of frames, e.g. bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames and a class of time-frequency localization operators.[24] The interest in g-frames arises from the fact that they provide more choices on analyzing functions than frame expansion coefficients[23] and also every fusion frame is a g-frame[5,7]". Generalized translation invariant (GTI) frames can be realized as g-frames,[18] so for motivating to answer the similar problems relevant to shift invariant and GTI systems in Ref. 8, we generalize some results of the frame representations with bounded operators in Refs. 9 and 12 to g-frames. Now, we summarize some facts about g-frames from Refs. 21 and 23. For more on related subjects to g-frames, we refer to Refs. 16, 19 and 20.

Definition 1.3. We say that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is a generalized frame, or simply g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ if there are two constants $0 < A_\Lambda \le B_\Lambda < \infty$ such that

$$A_\Lambda \|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B_\Lambda \|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

We call A_Λ, B_Λ the lower and upper g-frame bounds, respectively. Λ is called a tight g-frame if $A_\Lambda = B_\Lambda$, and a Parseval g-frame if $A_\Lambda = B_\Lambda = 1$. If for each $i \in I$, $\mathcal{K}_i = \mathcal{K}$, then, Λ is called a g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Note that for a family $\{\mathcal{K}_i\}_{i\in I}$ of Hilbert spaces, there exists a Hilbert space $\mathcal{K} = \oplus_{i\in I}\mathcal{K}_i$ such that for all $i \in I$, $\mathcal{K}_i \subseteq \mathcal{K}$, where $\oplus_{i\in I}\mathcal{K}_i$ is the direct sum of $\{\mathcal{K}_i\}_{i\in I}$. A family Λ is called a g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ if the right-hand inequality in (1.2) holds for all $f \in \mathcal{H}$, in this case, B_Λ is called a g-Bessel bound.

If there is no confusion, we use g-frame (g-Bessel family) instead of g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ (g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$).

Example 1.4 (Ref. 23). Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Suppose that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in I\}$, where

$$\Lambda_i f = \langle f, f_i\rangle, \quad f \in \mathcal{H}.$$

It is easy to see that Λ is a g-frame.

For a g-frame Λ, there exists a unique positive and invertible operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ such that

$$S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H},$$

and $A_\Lambda.\mathrm{Id}_{\mathcal{H}} \leq S_\Lambda \leq B_\Lambda.\mathrm{Id}_{\mathcal{H}}$. Consider the space

$$\left(\sum_{i\in I} \oplus \mathcal{K}_i\right)_{l^2} = \left\{\{g_i\}_{i\in I} : g_i \in \mathcal{K}_i, i \in I \text{ and } \sum_{i\in I} \|g_i\|^2 < \infty\right\}.$$

It is clear that $(\sum_{i\in I} \oplus \mathcal{K}_i)_{l^2}$ is a Hilbert space with pointwise operations and with the inner product given by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

For a g-Bessl family Λ, the synthesis operator $T_\Lambda : (\sum_{i\in I} \oplus \mathcal{K}_i)_{l^2} \to \mathcal{H}$ is defined by

$$T_\Lambda(\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i.$$

The adjoint of T_Λ, $T_\Lambda^* : \mathcal{H} \to (\sum_{i\in I} \oplus \mathcal{K}_i)_{l^2}$ is called the analysis operator of Λ and is as follows:

$$T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}, \quad f \in \mathcal{H}.$$

It is obvious that $S_\Lambda = T_\Lambda T_\Lambda^*$.

Definition 1.5. Two g-frames Λ and Θ are called dual if

$$\sum_{i\in I} \Lambda_i^* \Theta_i f = f, \quad f \in \mathcal{H}.$$

For a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$, the g-frame $\widetilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is a dual of Λ, which is called the canonical dual.

Definition 1.6. Consider a family $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$.

(i) We say that Λ is g-complete if $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$.
(ii) We say that Λ is a g-Riesz basis if Λ is g-complete and there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that for any finite set $\{g_i\}_{i\in I_n}$,

$$A_\Lambda \sum_{i\in I_n} \|g_i\|^2 \leq \|\sum_{i\in I_n} \Lambda_i^* g_i\|^2$$
$$\leq B_\Lambda \sum_{i\in I_n} \|g_i\|^2, \quad g_i \in \mathcal{K}_i.$$

(iii) We say that Λ is a g-orthonormal basis if it satisfies the following:

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in I, \ \ g_i \in \mathcal{K}_i, \ \ g_j \in \mathcal{K}_j,$$

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

Theorem 1.7 (Ref. 23). *A family* $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ *is a g-Riesz basis if and only if there exist a g-orthonormal basis* Θ *and* $U \in GL(\mathcal{H})$ *such that* $\Lambda_i = \Theta_i U, i \in I$.

Theorem 1.8 (Ref. 23). *Let for* $i \in I$, $\{e_{i,j}\}_{j \in J_i}$ *be an orthonormal basis for* $\mathcal{K}_i$.

(i) Λ *is a g-frame* (*respectively*, *g-Bessel family*, *g-Riesz basis*, *g-orthonormal basis*) *if and only if* $\{\Lambda_i^* e_{i,j}\}_{i \in I, j \in J_i}$ *is a frame* (*respectively*, *Bessel sequence*, *Riesz basis*, *orthonormal basis*).

(ii) Λ *and* Θ *are dual if and only if* $\{\Lambda_i^* e_{i,j}\}_{i \in I, j \in J_i}$ *and* $\{\Theta_i^* e_{i,j}\}_{i \in I, j \in J_i}$ *are dual.*

In this paper, we generalize some recent results of Christensen *et al.*[9,12] to investigate representations for g-frames with bounded operators.

2. Representations of *G*-Frames

In this section, by generalizing some results of Refs. 9 and 12, we introduce representations for g-frames with bounded operators and give some examples of g-frames with a representation and without any representations. In Theorem 2.5, we get sufficient conditions for g-frames to have a representation with a bounded operator. Also, Theorem 2.5 and Proposition 2.10 show that for g-frames $\Lambda = \{\Lambda_1 T^{i-1} : i \in \mathbb{N}\}$, the boundedness of T is equivalent to the invariance of $\ker T_\Lambda$ under the right-shift operator.

Remark 2.1. Consider a frame $F = \{f_i\}_{i \in \mathbb{N}} = \{T^{i-1} f_1\}_{i \in \mathbb{N}}$ for $\mathcal{H}$ with $T \in B(\mathcal{H})$. For the g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\}$ where

$$\Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H},$$

we have

$$\Lambda_{i+1} f = \langle f, f_{i+1} \rangle = \langle f, T f_i \rangle = \langle T^* f, f_i \rangle = \Lambda_i T^* f, \quad f \in \mathcal{H}.$$

Therefore, $\Lambda_i = \Lambda_1 (T^*)^{i-1}, i \in \mathbb{N}$. Conversely, if we consider a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\} = \{\Lambda_1 T^{i-1} : i \in \mathbb{N}\}$ for $T \in B(\mathcal{H})$, then by the Riesz representation theorem, $\Lambda_i f = \langle f, f_i \rangle, i \in \mathbb{N}$ and $f, f_i \in \mathcal{H}$, where $F = \{f_i\}_{i \in \mathbb{N}}$ is a frame such that $f_i = (T^*)^{i-1} f_1, i \in \mathbb{N}$.

Now, we are motivated to study g-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$, where $\Lambda_i = \Lambda_1 T^{i-1}$ with $T \in B(\mathcal{H})$.

Definition 2.2. We say that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ has a representation if there is a $T \in B(\mathcal{H})$ such that $\Lambda_i = \Lambda_1 T^{i-1}, i \in \mathbb{N}$. In the affirmative case, we say that Λ is represented by T.

In the following, we give some g-frames that have a representation.

Example 2.3. (i) The g-frame of finite elements $\Lambda = \{\Lambda_i \in GL(\mathcal{H}) : i = 1, 2\}$ is represented by $\Lambda_1^{-1}\Lambda_2$.

(ii) The tight g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}) : i \in \mathbb{N}\}$ with $\Lambda_i = \frac{2^{i-1}}{3^{i-2}}\mathrm{Id}_{\mathcal{H}}$ is represented by $\frac{2}{3}\mathrm{Id}_{\mathcal{H}}$.

(iii) Let $F = \{f_i\}_{i\in\mathbb{N}} = \{T^{i-1}f_1\}_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$, where $T \in B(\mathcal{H})$. Then the g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}^2) : i \in \mathbb{N}\}$ with $\Lambda_i f = (\langle f, f_i\rangle, \langle f, f_{i+1}\rangle)$, $f \in \mathcal{H}$, is represented by T^*.

Now, we give a g-frame without any representations.

Example 2.4. Consider the tight g-frame $\Lambda = \{\Lambda_n \in B(\mathbb{C}) : n \in \mathbb{N}\}$ with $\Lambda_n = \frac{1}{n^4+1}\mathrm{Id}_{\mathbb{C}}$. Since $\Lambda_1 = \frac{1}{2}\mathrm{Id}_{\mathbb{C}}$ and $\Lambda_2 = \frac{1}{17}\mathrm{Id}_{\mathbb{C}}$, the g-frame Λ has not any representations.

By generalizing a result of Ref. 10, the following theorem gives sufficient conditions for a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ to have a representation.

Theorem 2.5. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be a g-frame such that for every finite set $\{g_i\}_{i\in I_n} \subset \mathcal{K}$, $\sum_{i\in I_n} \Lambda_i^* g_i = 0$ implies $g_i = 0$ for every $i \in I_n$. Suppose that $\ker T_\Lambda$ is invariant under the right-shift operator. Then, Λ is represented by $T \in B(\mathcal{H})$, where $\|T\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$.*

Proof. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{K}$. We define the linear map $S : \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{N}} \to \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{N}}$ with

$$S(\Lambda_i^* e_j) = \Lambda_{i+1}^* e_j.$$

By the assumption, for any finite index sets $I_n \subset \mathbb{N}$ and $J_m \subset J$, $\sum_{i\in I_n, j\in J_m} c_{ij}\Lambda_i^* e_j = \sum_{i\in I_n} \Lambda_i^*(\sum_{j\in J_m} c_{ij}e_j) = 0$ implies $\sum_{j\in J_m} c_{ij}e_j = 0$ and so $c_{ij} = 0$ for $i \in I_n$, $j \in J_m$. Therefore, S is well-defined. Now, we show that S is bounded. Let $f = \sum_{i\in\mathbb{N}, j\in J} c_{ij}\Lambda_i^* e_j$ for $c_{ij} \in \ell^2(\mathbb{C}, \mathbb{N} \times J)$ with $c_{ij} = 0$, $i \notin I_n$ or $j \notin J_m$. By Theorem 1.8, $F = \{\Lambda_i^* e_j\}_{i\in\mathbb{N}, j\in J}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A_Λ and B_Λ, respectively. We can write $\{c_{ij}\}_{i\in\mathbb{N},j\in J} = \{d_{ij}\}_{i\in\mathbb{N},j\in J} + \{r_{ij}\}_{i\in\mathbb{N},j\in J}$ with $\{d_{ij}\}_{i\in\mathbb{N},j\in J} \in \ker T_F$ and $\{r_{ij}\}_{i\in\mathbb{N},j\in J} \in (\ker T_F)^\perp$. Since $\sum_{i\in\mathbb{N}} \Lambda_i^*\big(\sum_{j\in J} d_{ij}e_j\big) = \sum_{i\in\mathbb{N},j\in J} d_{ij}\Lambda_i^* e_j = 0$ and $\{\sum_{j\in J} d_{ij}e_j\}_{i\in\mathbb{N}} \in \ker T_\Lambda$, then by the assumption, we conclude that

$$\sum_{i\in\mathbb{N},j\in J} d_{ij}\Lambda_{i+1}^* e_j = \sum_{i\in\mathbb{N}} \Lambda_{i+1}^* \left(\sum_{j\in J} d_{ij}e_j\right) = 0,$$

and so the same as in the proof of Ref. 12, we have

$$\|Sf\|^2 = \left\|\sum_{i\in\mathbb{N},j\in J} c_{ij}\Lambda_{i+1}^* e_j\right\|^2 = \left\|\sum_{i\in\mathbb{N},j\in J} r_{ij}\Lambda_{i+1}^* e_j\right\|^2 \le B_\Lambda \sum_{i\in\mathbb{N},j\in J} |r_{ij}|^2. \tag{2.1}$$

Since $\{r_{ij}\}_{i\in\mathbb{N},j\in J} \in (\ker T_F)^{\perp}$, by Lemma 5.5.5 of Ref. 7, we have

$$A_\Lambda \sum_{i\in\mathbb{N},j\in J} |r_{ij}|^2 \leq \left\| \sum_{i\in\mathbb{N},j\in J} r_{ij}\Lambda_i^* e_j \right\|^2. \tag{2.2}$$

Therefore, by the inequalities (2.1) and (2.2), we have

$$\begin{aligned}\|Sf\|^2 &\leq B_\Lambda A_\Lambda^{-1} \left\| \sum_{i\in\mathbb{N},j\in J} r_{ij}\Lambda_i^* e_j \right\|^2 = B_\Lambda A_\Lambda^{-1} \left\| \sum_{i\in\mathbb{N},j\in J} (d_{ij}+r_{ij})\Lambda_i^* e_j \right\|^2 \\ &= B_\Lambda A_\Lambda^{-1} \left\| \sum_{i\in\mathbb{N},j\in J} c_{ij}\Lambda_i^* e_j \right\|^2 \\ &= B_\Lambda A_\Lambda^{-1}\|f\|^2.\end{aligned}$$

So, S is bounded and can be extended to $\bar{S} \in B(\mathcal{H})$. It is obvious that Λ is represented by $T = (\bar{S})^*$ and $\|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}$. In fact, for every $g \in \mathcal{K}$, we have

$$\begin{aligned}\bar{S}\Lambda_i^* g &= \bar{S}\Lambda_i^* \left(\sum_{j\in J} c_j e_j \right) = \sum_{j\in J} c_j \bar{S}\Lambda_i^* e_j \\ &= \sum_{j\in J} c_j S\Lambda_i^* e_j = \sum_{j\in J} c_j \Lambda_{i+1}^* e_j \\ &= \Lambda_{i+1}^* \left(\sum_{j\in J} c_j e_j \right) = \Lambda_{i+1}^* g, \quad i \in \mathbb{N}.\end{aligned}$$

□

Corollary 2.6. *Every g-orthonormal basis has a representation.*

Proof. For every finite set $\{g_i\}_{i\in I_n} \subset \mathcal{K}$, we have

$$\begin{aligned}\left\| \sum_{i\in I_n} \Lambda_i^* g_i \right\|^2 &= \left\langle \sum_{i\in I_n} \Lambda_i^* g_i, \sum_{j\in I_n} \Lambda_j^* g_j \right\rangle = \sum_{i\in I_n}\sum_{j\in I_n} \langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle \\ &= \sum_{i\in I_n} \langle g_i, g_i \rangle = \sum_{i\in I_n} \|g_i\|^2.\end{aligned}$$

So $\sum_{i\in I_n} \Lambda_i^* g_i = 0$ implies $g_i = 0$ for any $i \in I_n$. Similarly, we have $\ker T_\Lambda = \{0\}$, that is invariant under the right-shift operator. Then, by Theorem 2.5 the proof is completed. □

Remark 2.7. Consider a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ which is represented by T. For $S \in GL(\mathcal{H})$, the family $\Lambda S = \{\Lambda_i S \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ is a g-frame (Ref. 20, Corollary 2.26), which is represented by $S^{-1}TS$.

Corollary 2.8. *Every g-Riesz basis has a representation.*

Proof. By Theorem 1.7, Corollary 2.6 and Remark 2.7, the proof is completed. □

Now, we give an example to show that the converse of Theorem 2.5 is not satisfied.

Example 2.9. Consider the tight g-frame $\Lambda = \{\Lambda_i \in B(l^2(\mathcal{H},\mathbb{N})) : i \in \mathbb{N}\}$ with $\Lambda_i = (\frac{1}{2})^{i-1}\mathrm{Id}_{l^2(\mathcal{H},\mathbb{N})}$. It is obvious that Λ is represented by $\frac{1}{2}\mathrm{Id}_{l^2(\mathcal{H},\mathbb{N})}$, but $\Lambda_1^*(\frac{1}{2}e_1)+\Lambda_2^*(-e_1) = 0$ for $e_1 = (1, 0, 0, \ldots)$.

Proposition 2.10. *Let a g-frame* $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ *be represented by* T*. Then* $\ker T_\Lambda$ *is invariant under the right-shift operator* $\mathcal{T}$*.*

Proof. For any $\{g_i\}_{i\in\mathbb{N}} \in \ker T_\Lambda$, we have

$$T_\Lambda \mathcal{T}\{g_i\}_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}} \Lambda_{i+1}^* g_i = \sum_{i\in\mathbb{N}} T^*\Lambda_i^* g_i = T^*\left(\sum_{i\in\mathbb{N}} \Lambda_i^* g_i\right) = 0.$$ □

The following proposition shows that the converse of Theorem 2.5 is satisfied for one-dimensional Hilbert space $\mathcal{K}$.

Proposition 2.11. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be infinite-dimensional and one-dimensional Hilbert spaces, respectively, and a g-frame* $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ *be represented by* T*. Hence,* $\sum_{i\in I_n} \Lambda_i^* g_i = 0$ *implies* $g_i = 0$*, for any finite set* $\{g_i\}_{i\in I_n} \subset \mathcal{K}$*.*

Proof. Let $\{e_1\}$ be a basis for $\mathcal{K}$. By Theorem 1.8, the sequence $F = \{\Lambda_i^* e_1\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$. Since the g-frame Λ is represented by T, the frame F is represented by T^*, i.e.

$$\Lambda_i^* e_1 = (T^*)^{i-1}\Lambda_1^* e_1,$$

and so by Proposition 1.1, F is linearly independent. We have

$$0 = \sum_{i\in I_n} \Lambda_i^* g_i = \sum_{i\in I_n} \Lambda_i^*(\alpha_i e_1) = \sum_{i\in I_n} \alpha_i \Lambda_i^* e_1, \quad \alpha_i \in \mathbb{C},$$

therefore, for any $i \in I_n$, $\alpha_i = 0$ and so $g_i = 0$. □

Remark 2.12. Proposition 2.11 shows that for one-dimensional Hilbert space $\mathcal{K}$ with basis $\{e_1\}$, when a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ has a representation, then the frame $\{\Lambda_i^* e_1\}_{i\in\mathbb{N}}$ has a representation. For a finite-dimensional Hilbert space $\mathcal{K}$ with orthonormal basis $\{e_j\}_{j=1}^n$, when a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ is represented by T, then the frame $F = \{\Lambda_i^* e_j, j = 1, \ldots, n\}_{i\in\mathbb{N}}$ can be represented by T^* and finite vectors $\{\Lambda_1^* e_1, \ldots, \Lambda_1^* e_n\}$, i.e. $F = \{(T^*)^{i-1}\Lambda_1^* e_j, j = 1, \ldots, n\}_{i\in\mathbb{N}}$, then it can be worked on g-frames that be represented by a bounded operator and finite subset of the g-frame. But, Example 2.9 shows that for infinite-dimensional Hilbert space $K = l^2(\mathcal{H},\mathbb{N})$ with orthonormal basis $\{e_j\}_{j\in I}$, this may not happen, i.e. a g-frame Λ has a representation and the frame $\{\Lambda_i^* e_j\}_{i,j\in\mathbb{N}}$ does not have. By Theorem 1.8, for a g-Riesz basis $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$, the sequence $F = \{\Lambda_i^* e_j\}_{i\in\mathbb{N}, j\in I}$ is a Riesz basis. By Corollary 2.8 and [12, Example 2.2], both of Λ and F have representations. What is the relation between these two representations (open problem)?

Now, we want to discuss the concept of representation for g-frames with index set $\mathbb{Z}$.

Definition 2.13. We say that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ has a representation if there is a $T \in GL(\mathcal{H})$ such that $\Lambda_i = \Lambda_0 T^i, i \in \mathbb{Z}$. In the affirmative case, we say that Λ is represented by T.

Example 2.14. Consider the tight g-frame $\Lambda = \{\Lambda_n \in B(\mathbb{C}) : n \in \mathbb{Z}\}$ with $\Lambda_n = \frac{1}{n^2-2n+4}\mathrm{Id}_{\mathbb{C}}$. Since $\Lambda_1 = \frac{1}{3}\mathrm{Id}_{\mathbb{C}}$ and $\Lambda_3 = \frac{1}{7}\mathrm{Id}_{\mathbb{C}}$, the g-frame Λ has not any representation.

A subspace $V \subseteq l^2(\mathcal{H},\mathbb{Z})$ is said invariant under the right-shift (left-shift) operator if $\mathcal{T}(V) \subseteq V$ $(\mathcal{T}^*(V) \subseteq V)$.

Theorem 2.15. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ be a g-frame such that for every finite set $\{g_i\}_{i\in I_n} \subset \mathcal{K}$, $\sum_{i\in I_n} \Lambda_i^* g_i = 0$ implies $g_i = 0$ for every $i \in I_n$. Suppose that $\ker T_\Lambda$ is invariant under the right-shift and left-shift operators. Then, Λ is represented by $T \in GL(\mathcal{H})$, where $\|T\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$.*

Proof. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{K}$. We define the linear map $S : \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}} \to \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}}$ with

$$S(\Lambda_i^* e_j) = \Lambda_{i+1}^* e_j.$$

Similar to the proof of the Theorem 2.5, S is well-defined and bounded with $\|S\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$. Consider the linear map $S^{-1} : \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}} \to \mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}}$ with

$$S^{-1}(\Lambda_i^* e_j) = \Lambda_{i-1}^* e_j.$$

Similar to S, the map S^{-1} is also well-defined and since $\ker T_\Lambda$ is invariant under the left-shift operator, S^{-1} is bounded. It is obvious that $SS^{-1} = S^{-1}S = \mathrm{Id}_{\mathrm{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}}}$. The operators S and S^{-1} can be extended on $\mathcal{H}$. It is obvious that Λ is represented by $T = (\bar{S})^*$, where $\bar{S} \in GL(\mathcal{H})$ is the extension of S and $\|T\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$. □

Remark 2.16. Note that if $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ is a g-orthonormal basis or g-Riesz basis, then by Theorem 2.15, Λ has a representation.

Theorem 2.17. *Let a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ be represented by T, then $\ker T_\Lambda$ is invariant under the right-shift and left-shift operators and*

$$1 \le \|T\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}, \quad 1 \le \|T^{-1}\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}.$$

Proof. Similar to Proposition 2.10, $\ker T_\Lambda$ is invariant under the right-shift operator. Also for $\{g_i\}_{i\in\mathbb{Z}} \in \ker T_\Lambda$,

$$\begin{aligned} T_\Lambda \mathcal{T}^* \{g_i\}_{i\in\mathbb{Z}} &= \sum_{i\in\mathbb{Z}} \Lambda_{i-1}^* g_i = \sum_{i\in\mathbb{Z}} (T^{i-1})^* \Lambda_0^* g_i \\ &= (T^{-1})^* \left(\sum_{i\in\mathbb{Z}} (T^i)^* \Lambda_0^* g_i \right) \\ &= (T^{-1})^* \left(\sum_{i\in\mathbb{Z}} \Lambda_i^* g_i \right) \\ &= (T^{-1})^* T_\Lambda \{g_i\}_{i\in\mathbb{Z}} = 0. \end{aligned}$$

So, $\ker T_\Lambda$ is also invariant under the left-shift operator. Now for some fixed $n \in \mathbb{N}$ and $0 \neq f \in \mathcal{H}$ we have

$$\begin{aligned} A_\Lambda \|f\|^2 &\le \sum_{i\in\mathbb{Z}} \|\Lambda_i f\|^2 = \sum_{i\in\mathbb{Z}} \|\Lambda_0 T^i f\|^2 = \sum_{i\in\mathbb{Z}} \|\Lambda_0 T^i T^{-n} T^n f\|^2 \\ &= \sum_{i\in\mathbb{Z}} \|\Lambda_0 T^{i-n} T^n f\|^2 \\ &= \sum_{i\in\mathbb{Z}} \|\Lambda_i T^n f\|^2 \\ &\le B_\Lambda \|T^n f\|^2 \le B_\Lambda \|T\|^{2n} \|f\|^2, \end{aligned}$$

that implies $\|T\| \ge 1$. Since for any $i \in \mathbb{Z}$, $\Lambda_i T = \Lambda_{i+1}$, we have $T^* \Lambda_i^* e_j = \Lambda_{i+1}^* e_j$. So, T^* is the operator $\bar{S}$ that is defined in the proof of Theorem 2.5, just on $\operatorname{span}\{\Lambda_i^*(\mathcal{K})\}_{i\in\mathbb{Z}}$ and therefore we have $\|T\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$, alike. Since $\Lambda = \{\Lambda_{-i} \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\} = \{\Lambda_0 (T^{-1})^i : i \in \mathbb{Z}\}$, by replacing T^{-1} instead of T, we get $1 \le \|T^{-1}\| \le \sqrt{B_\Lambda A_\Lambda^{-1}}$. □

Example 2.3, (ii) shows that for the index set $\mathbb{N}$, $1 \le \|T\|$ does not happen, in general.

Corollary 2.18. *Let a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ be represented by $T \in GL(\mathcal{H})$. Then the following hold*:

(i) *If Λ is a tight g-frame, then $\|T\| = \|T^{-1}\| = 1$ and so T is isometry.*
(ii) $\|S_\Lambda^{\frac{1}{2}} T S_\Lambda^{-\frac{1}{2}}\| = \|S_\Lambda^{\frac{1}{2}} T^{-1} S_\Lambda^{\frac{-1}{2}}\| = 1$.

The authors of Ref. 11 considered sequences in $\mathcal{H}$ of the form $F = \{T^i f_0\}_{i\in I}$, with a linear operator T to study for which bounded operator T and vector $f_0 \in \mathcal{H}$, F is a frame for $\mathcal{H}$. In Proposition 3.5 of Ref. 12, it was proved that if the operator $T \in B(\mathcal{H})$ is compact, then the sequence $\{T^i f_0\}_{i\in I}$ cannot be a frame for infinite-dimensional $\mathcal{H}$. Someone can study these results for family of operators $\{\Lambda_0 T^i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ for $T \in B(\mathcal{H})$ and $\Lambda_0 \in B(\mathcal{H},\mathcal{K})$.

3. Representations of Dual *G*-Frames

The purpose of this section is to get a necessary and sufficient condition for a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\}$ to have a representation, by applying the concept of duality. Also, for some g-frames with representation, we get a dual with representation and in one case without representation. In the end, we get the relation between representations of dual g-frames with index set $\mathbb{Z}$. The proofs of the results are similar to Refs. 9 and 12.

Theorem 3.1. *A g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is represented by T if and only if for a dual $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ of Λ (and hence all),*

$$\Lambda_{k+1} = \sum_{i\in\mathbb{N}} \Lambda_k \Theta_i^* \Lambda_{i+1}.$$

Proof. First, assume that Λ is represented by T. For any $g \in \mathcal{K}$ we have

$$\begin{aligned}\Lambda_{k+1}^* g = T^* \Lambda_k^* g = T^* \left(\sum_{i\in\mathbb{N}} \Lambda_i^* \Theta_i \Lambda_k^* g \right) \\ = \sum_{i\in\mathbb{N}} T^* \Lambda_i^* \Theta_i \Lambda_k^* g \\ = \sum_{i\in\mathbb{N}} \Lambda_{i+1}^* \Theta_i \Lambda_k^* g \\ = \sum_{i\in\mathbb{N}} (\Lambda_k \Theta_i^* \Lambda_{i+1})^* g \\ = \left(\sum_{i\in\mathbb{N}} \Lambda_k \Theta_i^* \Lambda_{i+1} \right)^* g,\end{aligned}$$

then, $\Lambda_{k+1} = \sum_{i\in\mathbb{N}} \Lambda_k \Theta_i^* \Lambda_{i+1}$.

Conversely, it is obvious that $\Lambda_k T = \Lambda_{k+1}$ for $Tf = \sum_{i\in\mathbb{N}} \Theta_i^* \Lambda_{i+1} f$. □

Remark 3.2. By Corollary 3.3 of Ref. 23, for a g-Riesz basis $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$, we have

$$\begin{aligned}\left\langle \sum_{i\in\mathbb{N}} \Lambda_k \widetilde{\Lambda}_i^* \Lambda_{i+1} f, g \right\rangle &= \sum_{i\in\mathbb{N}} \langle \widetilde{\Lambda}_i^* \Lambda_{i+1} f, \Lambda_k^* g \rangle \\ &= \sum_{i\in\mathbb{N}} \delta_{i,k} \langle \Lambda_{i+1} f, g \rangle = \langle \Lambda_{k+1} f, g \rangle, \quad f \in \mathcal{H}, \ g \in \mathcal{K},\end{aligned}$$

therefore, by Theorem 3.1, Λ has a representation.

In the following, we want to investigate that if a g-frame Λ has a representation, its duals have representations or not. If so, what is the relation between their representations?

Example 3.3. (i) Assume that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ is represented by T. Then, by Remark 2.7, the canonical dual $\widetilde{\Lambda}$ is represented by $S_\Lambda T S_\Lambda^{-1}$.

(ii) Consider the g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}) : i \in \mathbb{N}\}$ with $\Lambda_i = (\frac{2}{3})^i \mathrm{Id}_{\mathcal{H}}$, which is represented by $\frac{2}{3}\mathrm{Id}_{\mathcal{H}}$. The g-frame $\Theta = \{\Theta_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{N}\}$ with $\Theta_i = (\frac{3}{4})^i \mathrm{Id}_{\mathcal{H}}$ is a dual of Λ which is represented by $\frac{3}{4}\mathrm{Id}_{\mathcal{H}}$.

(iii) The g-frame of finite elements $\Lambda = \{\Lambda_i \in B(\mathbb{C}) : i = 1,2,3\}$ with $\Lambda_i = 2^{i-1}\mathrm{Id}_{\mathbb{C}}$ is represented by $2\mathrm{Id}_{\mathbb{C}}$, but the dual $\Theta = \{\Theta_i \in B(\mathbb{C}) : i = 1,2,3\}$ of Λ with $\Theta_1 = -2\mathrm{Id}_{\mathbb{C}}, \Theta_2 = \mathrm{Id}_{\mathbb{C}}$ and $\Theta_3 = \frac{1}{4}\mathrm{Id}_{\mathbb{C}}$ does not have any representation. Note that the dual $\Gamma = \{\Gamma_i \in B(\mathbb{C}) : i = 1,2,3\}$ of Λ with $\Gamma_i = \frac{1}{3}(\frac{1}{2})^{i-1}\mathrm{Id}_{\mathbb{C}}$ is represented by $\frac{1}{2}\mathrm{Id}_{\mathbb{C}}$.

Proposition 3.4. *Let a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\}$ be represented by $T \in GL(\mathcal{H})$. Then, the canonical dual $\widetilde{\Lambda}$ is represented by $S_\Lambda T S_\Lambda^{-1} = (T^*)^{-1}$.*

Proof. It is obvious that $\widetilde{\Lambda}$ is represented by $S_\Lambda T S_\Lambda^{-1}$. For any $\{g_i\}_{i\in\mathbb{Z}} \in l^2(\mathcal{K},\mathbb{Z})$,

$$T^*T_\Lambda\{g_i\}_{i\in\mathbb{Z}} = \sum_{i\in\mathbb{Z}} T^*\Lambda_i^* g_i = \sum_{i\in\mathbb{Z}} (\Lambda_i T)^* g_i = \sum_{i\in\mathbb{Z}} \Lambda_{i+1}^* g_i = T_\Lambda \mathcal{T}\{g_i\}_{i\in\mathbb{Z}}.$$

So, we have

$$T^*S_\Lambda T = T^*T_\Lambda T_\Lambda^* T = T^*T_\Lambda (T^*T_\Lambda)^* = T_\Lambda \mathcal{T}\mathcal{T}^* T_\Lambda^* = T_\Lambda T_\Lambda^* = S_\Lambda.$$

Therefore, $S_\Lambda T S_\Lambda^{-1} = (T^*)^{-1}$. □

Remark 3.5. Let $F = \{f_i\}_{i\in\mathbb{Z}}$ and $G = \{g_i\}_{i\in\mathbb{Z}}$ be dual frames that are represented by $T, S \in GL(\mathcal{H})$, respectively. Then, by Remark 2.1, the dual g-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathbb{C}) : i \in \mathbb{Z}\}$ with $\Lambda_i f = \langle f, f_i\rangle$ and $\Theta = \{\Theta_i \in B(\mathcal{H},\mathbb{C}) : i \in \mathbb{Z}\}$ with $\Theta_i f = \langle f, g_i\rangle$ are represented by $T^*, S^* \in GL(\mathcal{H})$, respectively. By Lemma 3.3 of Ref. 9, $S = (T^*)^{-1}$.

The relation between representations of dual g-frames by the index set $\mathbb{Z}$ is given in what follows.

Theorem 3.6. *Assume that $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\} = \{\Lambda_0 T^i : i \in \mathbb{Z}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H},\mathcal{K}) : i \in \mathbb{Z}\} = \{\Theta_0 S^i : i \in \mathbb{Z}\}$ are dual g-frames, where $T, S \in GL(\mathcal{H})$. Then, $S = (T^*)^{-1}$.*

Proof. For any $f \in \mathcal{H}$, we have

$$\begin{aligned} f &= \sum_{i\in\mathbb{Z}} \Lambda_i^*\Theta_i f = \sum_{i\in\mathbb{Z}} (T^*)^i \Lambda_0^*\Theta_0 S^i f \\ &= T^*\sum_{i\in\mathbb{Z}} (T^*)^{i-1}\Lambda_0^*\Theta_0 S^{i-1} S f = T^*\sum_{i\in\mathbb{Z}} \Lambda_i^*\Theta_i S f = T^*Sf. \end{aligned}$$

Since $T \in GL(\mathcal{H})$, the proof is completed. □

In general, Theorem 3.6 is not true for the index set $\mathbb{N}$ (see Example 3.1, (ii)).

4. Stability of G-Frame Representations

Christensen considered the stability of the frames in Hilbert spaces under perturbations.[6] Similar to ordinary frames, Sun proved that g-frames are stable under small perturbations and have studied the stability of dual g-frames.[24] You can find more perturbation results for g-frames in Ref. 20. In Ref. 9, we find a perturbation condition that preserves the existence of a representation for a frame. In this section, we study the stability of g-frame representations under some perturbations.

Theorem 4.1. *Suppose that a g-frame* $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$, $(I = \mathbb{N}$ *or* $\mathbb{Z})$ *has a representation and* $\Gamma = \{\Gamma_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ *is a family of operators such that for every finite set* $\{g_i\}_{i\in I_n} \subset \mathcal{K}$,

$$\left\|\sum_{i\in I_n}(\Lambda_i - \Gamma_i)^* g_i\right\| \leq \lambda \left\|\sum_{i\in I_n}\Lambda_i^* g_i\right\| + \mu \left\|\sum_{i\in I_n}\Gamma_i^* g_i\right\|, \tag{4.1}$$

where $0 \leq \max\{\lambda,\mu\} < 1$. *Then, the family* Γ *is a g-frame that has a representation.*

Proof. The family Γ is a g-frame (Ref. 20, Theorem 3.5). By the inequality (4.1), we get $\ker T_\Lambda = \ker T_\Gamma$. The operator T_Λ is onto (Ref. 20, Proposition 2.6) and so for any $f \in \mathcal{H}$, there is $\{a_i\}_{i\in I} \in l^2(\mathcal{K}, I)$ such that $T_\Lambda\{a_i\}_{i\in I} = f$. We define the well-defined operator $U \in B(\mathcal{H})$ by $Uf = T_\Gamma\{a_i\}_{i\in I}$. By the inequality (4.1), U is injective. On the other hand, T_Γ is also onto and so U is onto. Therefore, $U \in GL(\mathcal{H})$. For any $\{g_i\}_{i\in I} \in l^2(\mathcal{K}, I)$ and $g \in \mathcal{H}$,we have

$$\begin{aligned}
\langle\{g_i\}_{i\in I}, \{(\Gamma_i - \Lambda_i U^*)g\}_{i\in I}\rangle &= \sum_{i\in I}\langle g_i, (\Gamma_i - \Lambda_i U^*)g\rangle \\
&= \sum_{i\in I}\langle \Gamma_i^* g_i, g\rangle - \sum_{i\in I}\langle \Lambda_i^* g_i, U^* g\rangle \\
&= \langle T_\Gamma\{g_i\}_{i\in I}, g\rangle - \langle UT_\Lambda\{g_i\}_{i\in I}, g\rangle \\
&= \langle UT_\Lambda\{g_i\}_{i\in I}, g\rangle - \langle UT_\Lambda\{g_i\}_{i\in I}, g\rangle \\
&= 0,
\end{aligned}$$

therefore $\Gamma_i = \Lambda_i U^*, i \in I$ and so if Λ is represented by T, then Γ is represented by $(U^*)^{-1}TU^*$. Indeed, we have

$$\Gamma_i(U^*)^{-1}TU^* = \Lambda_i U^*(U^*)^{-1}TU^* = \Lambda_i TU^* = \Lambda_{i+1}U^* = \Gamma_{i+1}. \qquad \square$$

Proposition 4.2. *Suppose that a g-frame* $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$, $(I = \mathbb{N}$ *or* $\mathbb{Z})$ *has a representation and* $\Gamma = \{\Gamma_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ *is a g-frame such that for a constant* $C > 0$,

$$\left\|\sum_{i\in I}(\Lambda_i - \Gamma_i)^* g_i\right\|^2 \leq C.\min\left\{\left\|\sum_{i\in I}\Lambda_i^* g_i\right\|^2, \left\|\sum_{i\in I}\Gamma_i^* g_i\right\|^2\right\}, \tag{4.2}$$

for $\{g_i\}_{i\in I} \in l^2(\mathcal{K}, I)$. *Then,* Γ *has a representation.*

Proof. By the inequality (4.2), it is obvious that $\ker T_\Lambda = \ker T_\Gamma$. So, by the same argument as in the proof of Theorem 4.1, Θ has a representation. □

Corollary 4.3. *Suppose that a frame $F = \{f_i\}_{i\in I}$, ($I = \mathbb{N}$ or $\mathbb{Z}$) has a representation and $G = \{g_i\}_{i\in I}$ is a frame for $\mathcal{H}$ such that for a constant $C > 0$,*

$$\left\|\sum_{i\in I} c_i(f_i - g_i)\right\|^2 \leq C.\min\left\{\left\|\sum_{i\in I} c_i f_i\right\|^2, \left\|\sum_{i\in I} c_i g_i\right\|^2\right\}, \tag{4.3}$$

for $\{c_i\}_{i\in I} \in l^2(\mathcal{H}, I)$. Then, the frame G has a representation.

Proof. By Remark 2.1 and Proposition 4.2, the proof is obvious. □

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