FIBONACCI REPRESENTATIONS OF SEQUENCES IN HILBERT SPACES

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Dynamical sampling deals with frames of the form $\{T^n\varphi\}_{n=0}^{\infty}$, *where* $T \in B(\mathcal{H})$ *belongs to certain classes of linear operators and* $\varphi \in \mathcal{H}$. *The purpose of this paper is to investigate a new representation, namely, Fibonacci representation of sequences* $\{f_n\}_{n=1}^{\infty}$ *in a Hilbert space* $\mathcal{H}$*; having the form* $f_{n+2} = T(f_n + f_{n+1})$ *for all* $n \geqslant 1$ *and a linear operator* $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$. *We apply this kind of representations for complete sequences and frames. Finally, we present some properties of Fibonacci representation operators.*

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1. **Introduction**

The concept of frames (discrete frames) in Hilbert spaces has been introduced by Duffin and Schaefer [8] in 1952 to study some problems in non-harmonic Fourier series and this is the starting point of frame theory. A frame for a separable Hilbert space $\mathcal{H}$ is a family of vectors in $\mathcal{H}$ which provides robust, stable and usually non-unique representations of vectors in $\mathcal{H}$. Indeed, frames can be viewed as redundant bases which are generalization of orthonormal bases. Vectors in a Hilbert space $\mathcal{H}$ may have different representations each useful for solving a certain problem. Frames are useful in areas such as coding theory, communication theory, signal processing and sampling theory, among others.

We recall some definitions and standard results from frame theory.

Definition 1.1. Consider a sequence $F = \{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$.

(i) F is called a *frame* for $\mathcal{H}$, if there exist two constants $A_F, B_F > 0$ such that

$$A_F\|f\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leqslant B_F\|f\|^2, \quad f \in \mathcal{H}.$$

(ii) F is called a *Bessel sequence* with Bessel bound B_F if at least the upper frame condition holds.

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(*iii*) F is called *complete* in $\mathcal{H}$ if $\overline{\text{span}}\{f_i\}_{i=1}^{\infty} = \mathcal{H}$, i.e., $\text{span}\{f_i\}_{i=1}^{\infty}$ is dense in $\mathcal{H}$.

(*iv*) F is called *linearly independent* if $\sum_{k=1}^{m} c_k f_k = 0$ for all $m \in \mathbb{N}$ and some scalar coefficients $\{c_k\}_{k=1}^{m}$, then $c_k = 0$ for all $k = 1, \cdots, m$. We say F is *linearly dependent* if F is not linearly independent.

Theorem 1.1. [4, Theorem 5.5.1] *A sequence $F = \{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if*

$$T_F : \ell^2 \to \mathcal{H}, \quad T_F(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i,$$

is a well-defined mapping from ℓ^2 onto $\mathcal{H}$. Moreover, the adjoint of T_F is given by

$$T_F^* : \mathcal{H} \to \ell^2, \quad T_F^* f = \{\langle f, f_i\rangle\}_{i=1}^{\infty}.$$

In [2], Aldroubi et al. introduced dynamic sampling which it deals with frame properties of sequences of the form $\{T^n\varphi\}_{n=0}^{\infty}$, where $T \in B(\mathcal{H})$ belongs to certain classes of linear operators (such as diagonalizable normal operators) and $\varphi \in \mathcal{H}$. Various characterizations of frames having the form $\{f_k\}_{k\in I} = \{T^k\varphi\}_{k\in I}$, where T is a linear (not necessarily bounded) operator can be found in [1, 3, 5, 6, 7, 9].

Proposition 1.1. [6, Proposition 2.3] *Consider a frame sequence $F = \{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ which spans an infinite-dimensional subspace. The following are equivalent:*

(*i*) *F is linearly independent.*

(*ii*) *There exists a linear operator $T : \text{span}\{f_i\}_{i=1}^{\infty} \to \mathcal{H}$ such that $Tf_i := f_{i+1}$.*

Theorem 1.2. [7, Theorem 2.1] *Consider a frame $F = \{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$. Then the following are equivalent:*

(*i*) *$F = \{T^{i-1}f_1\}_{i=1}^{\infty}$ for some $T \in B(\mathcal{H})$.*

(*ii*) *The $\ker T_F$ is invariant under the right-shift operator $\mathcal{T} : \ell^2 \to \ell^2$ defined by $\mathcal{T}(c_1, c_2, \cdots) = (0, c_1, c_2, \cdots)$.*

2. **Special sequences**

It is well known, cf. [4, Example 5.4.6] that if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, then $\{e_n+e_{n+1}\}_{n=1}^{\infty}$ is complete and a Bessel sequence but not a frame. This motivates us to investigate some results concerning the sequences $F = \{f_n\}_{n=1}^{\infty}$, $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ and $N = \{f_n - f_{n-1}\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$.

Proposition 2.1. *Let α and β be nonzero scalars and $F = \{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$. Then*

(*i*) *F is a Bessel sequence for $\mathcal{H}$, if and only if $M = \{\alpha f_n + \beta f_{n+1}\}_{n=1}^{\infty}$ and $N = \{\alpha f_n - \beta f_{n+1}\}_{n=1}^{\infty}$ are Bessel sequences for $\mathcal{H}$.*

(*ii*) *Suppose that F is a Bessel sequence for $\mathcal{H}$. Then F is complete, if and only if $M = \{\alpha f_n + \beta f_{n+1}\}_{n=1}^{\infty}$ is complete, whenever $|\alpha| \geqslant |\beta|$.*

Proof. (*i*) Assume that $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence with Bessel bound B_F and $\mu = \max\{|\alpha|^2, |\beta|^2\}$. Then for $f \in \mathcal{H}$, we have

$$\sum_{n=1}^{\infty} |\langle f, \alpha f_n + \beta f_{n+1}\rangle|^2 + \sum_{n=1}^{\infty} |\langle f, \alpha f_n - \beta f_{n+1}\rangle|^2 \leqslant 4\mu B_F \|f\|^2.$$

Then M and N are Bessel sequences. For the opposite implication, let B_M and B_N be Bessel bounds for sequences M and N, respectively. Then

$$2|\alpha|^2 \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant (B_M + B_N)\|f\|^2, \quad f \in \mathcal{H}.$$

(ii) Suppose that F is complete and $f \in \mathcal{H}$ such that $\langle f, \alpha f_n + \beta f_{n+1}\rangle = 0$ for all $n \in \mathbb{N}$. Then $\overline{\alpha}\langle f, f_n\rangle = -\overline{\beta}\langle f, f_{n+1}\rangle$ for all $n \in \mathbb{N}$. Since $|\alpha| \geqslant |\beta|$ and

$$|\langle f, f_1\rangle|^2 \sum_{n=0}^{\infty} \left|\frac{\alpha}{\beta}\right|^{2n-2} = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant B_F\|f\|^2,$$

we get $\langle f, f_1\rangle = 0$ and consequently $\langle f, f_n\rangle = 0$ for $n \in \mathbb{N}$. Hence $f = 0$ and this shows that $\{\alpha f_n + \beta f_{n+1}\}_{n=1}^{\infty}$ is complete. In order to show the other implication, assume that M is complete and $f \in \mathcal{H}$ such that $\langle f, f_n\rangle = 0$ for all $n \in \mathbb{N}$. Since

$$\langle f, \alpha f_n + \beta f_{n+1}\rangle = \overline{\alpha}\langle f, f_n\rangle + \overline{\beta}\langle f, f_{n+1}\rangle = 0, \quad n \in \mathbb{N},$$

we conclude that $f = 0$ and therefore F is complete.

□

Proposition 2.2. *Let $F = \{f_n\}_{n=1}^{\infty}$, $M = \{\alpha f_n + \beta f_{n+1}\}_{n=1}^{\infty}$ and $N = \{\alpha f_n - \beta f_{n+1}\}_{n=1}^{\infty}$ be sequences in a Hilbert space $\mathcal{H}$ and $\alpha \neq 0$. Then F is a frame for $\mathcal{H}$, if and only if $M \cup N$ is a frame for $\mathcal{H}$.*

Proof. Let $\mu = \max\{|\alpha|^2, |\beta|^2\}$. Then the result follows from

$$|\alpha|^2 \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant 4\mu \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2, \quad f \in \mathcal{H}.$$

□

Theorem 2.1. *Let $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ and $N = \{f_n - f_{n+1}\}_{n=1}^{\infty}$ be frames for $\mathcal{H}$. Then $F = \{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ and*

$$4S_F f = S_M f + S_N f + 2\langle f, f_1\rangle f_1, \quad f \in \mathcal{H}, \tag{1}$$

where S_F, S_M and S_N are frame operators for F, M and N, respectively.

Proof. By Proposition 2.1, F is a Bessel sequence for $\mathcal{H}$. Let A_M and A_N be lower frame bounds for M and N, respectively. Then we have

$$(A_M + A_N)\|f\|^2 \leqslant 4 \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2, \quad f \in \mathcal{H}.$$

Therefore, F is a frame for $\mathcal{H}$. Furthermore, since for $f \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} |\langle f, f_n + f_{n+1}\rangle|^2 + \sum_{n=1}^{\infty} |\langle f, f_n - f_{n+1}\rangle|^2 = 2\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 + 2\sum_{n=1}^{\infty} |\langle f, f_{n+1}\rangle|^2,$$

we obtain (1), by

$$\langle S_M f, f\rangle + \langle S_N f, f\rangle = 4\langle S_F f, f\rangle - 2\big\langle \langle f, f_1\rangle f_1, f\big\rangle, \quad f \in \mathcal{H}.$$

□

Theorem 2.2. *Let* $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$, $N = \{f_n - f_{n+1}\}_{n=1}^{\infty}$ *and* $F = \{f_n\}_{n=1}^{\infty}$ *be Bessel sequences for* $\mathcal{H}$ *and* $\ker T_F$ *be invariant under* $\mathcal{T}_R$. *Then* $\ker T_F = \ker T_M \cap \ker T_N$.

Proof. Let $\{c_n\}_{n=1}^{\infty} \in \ker T_F$. Since $\ker T_F$ is invariant under $\mathcal{T}_R$, we get $\{0, c_1, c_2, ...\} \in \ker T_F$. Therefore $\{c_1, c_1 + c_2, c_2 + c_3, ...\}, \{c_1, c_2 - c_1, c_3 - c_2, ...\} \in \ker T_F$. Hence

$$\sum_{n=1}^{\infty} c_n(f_n - f_{n+1}) = \sum_{n=1}^{\infty} c_n(f_n + f_{n+1}) = \sum_{n=1}^{\infty}(c_n + c_{n+1})f_{n+1} + c_1 f_1 = 0.$$

Then we conclude $\{c_n\}_{n=1}^{\infty} \in \ker T_M \cap \ker T_N$. On the other hand, if $\{c_n\}_{n=1}^{\infty} \in \ker T_M \cap \ker T_N$, then we have

$$0 = \sum_{n=1}^{\infty} c_n(f_n - f_{n+1}) + \sum_{n=1}^{\infty} c_n(f_n + f_{n+1}) = 2\sum_{n=1}^{\infty} c_n f_n.$$

Therefore, $\{c_n\}_{n=1}^{\infty} \in \ker T_F$. □

3. **Fibonacci representation**

In this section we want to consider representation of a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ on the form $f_n = T(f_{n-1} + f_{n-2})$ for $n \geqslant 3$, where T is a linear operator defined on an appropriate subspace of $\mathcal{H}$.

Definition 3.1. We say that a sequence $F = \{f_n\}_{n=1}^{\infty}$ has a *Fibonacci representation* if there is a linear operator $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ such that $f_n = T(f_{n-1} + f_{n-2})$ for $n \geqslant 3$. In the affirmative case, we say that F is represented by T, and T is called a *Fibonacci representation operator* with respect to F.

Throughout this segment, $\mathcal{H}$ denotes a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$.

Example 3.1. *It is clear that* $F = \{f_n\}_{n=1}^{\infty} = \{e_1, e_1, e_2, ...\}$ *is a frame for* $\mathcal{H}$. *We define the linear operator* $T : \operatorname{span}\{e_n\}_{n=1}^{\infty} \to \operatorname{span}\{e_n\}_{n=1}^{\infty}$ *by*

$$Te_1 = \frac{e_2}{2}, \quad Te_n = \sum_{i=0}^{n-2}(-1)^i e_{n-i+1} + (-1)^{n+1}\frac{e_2}{2}, \quad n \geqslant 2.$$

Then F *is represented by* T. *Note that* F *is not linearly independent, and so by* [6, Proposition 2.3], *there does not exist a linear operator* $S : \operatorname{span}\{e_n\}_{n=1}^{\infty} \to \operatorname{span}\{e_n\}_{n=1}^{\infty}$ *such that* $Se_1 = e_1$ *and* $Se_{n-1} = e_n$, $n \geqslant 2$.

Example 3.2. *The frame* $F = \{f_n\}_{n=1}^{\infty} = \{e_1, e_2, e_3, e_1, e_4, e_5, e_6, ...\}$ *is represented by* T, *where* $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ *is defined by*

$$Te_1 = \frac{1}{2}(e_4 + e_3 - e_1), \quad Te_2 = \frac{1}{2}(-e_4 + e_3 + e_1), \quad Te_3 = \frac{1}{2}(e_4 - e_3 + e_1),$$
$$Te_4 = e_5 - Te_1, \quad Te_n = e_{n+1} - Te_{n-1}, \quad n \geqslant 5.$$

Proposition 3.1. *A sequence* $F = \{f_n\}_{n=1}^{\infty}$ *is represented by* T, *if and only if* $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ *and* $N = \{f_n - f_{n+1}\}_{n=1}^{\infty}$ *are represented by* T.

Proof. First, let F be represented by T. For every $n \in \mathbb{N}$, we have

$$T\big((f_n \pm f_{n+1}) + (f_{n+1} \pm f_{n+2})\big) = f_{n+2} \pm f_{n+3},$$

Then M and N are represented by T. Conversely, if M and N are represented by T, then for all $n \in \mathbb{N}$, we have

$$T(f_n + f_{n+1}) = \frac{1}{2}T(f_n + f_{n+1} + f_n - f_{n+1} + f_{n+1} + f_{n+2} + f_{n+1} - f_{n+2}) = f_{n+2}.$$

Hence F is represented by T. □

A frame may have more than one Fibonacci representation and a frame may not have any.

Example 3.3. *The frame $G = \{f_n\}_{n=1}^{\infty} = \{e_1, e_2, e_1, e_3, e_4, ...\}$ does not have any Fibonacci representations. Indeed, if G is represented by T, then*

$$Te_1 + Te_2 = e_1, \quad Te_2 + Te_1 = e_3,$$

which is a contradiction. We note that $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ is not linearly independent.

Example 3.4. *Consider the frame $E = \{e_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ and let $T, S : \operatorname{span}\{e_n\}_{n=1}^{\infty} \to \operatorname{span}\{e_n\}_{n=1}^{\infty}$ be linear operators defined by*

$$Te_1 = Te_2 = \frac{1}{2}e_3, \quad Te_n = \frac{(-1)^n}{2}e_3 - \sum_{i=4}^{n+1}(-1)^{n+i-1}e_i, \quad n \geqslant 3,$$

$$Se_1 = 0,\ Se_2 = e_3,\ Se_3 = e_4 - e_3, \quad Se_n = e_3 - \sum_{i=4}^{n+1}(-1)^{n+i}e_i, \quad n \geqslant 4.$$

Then it is easy to see that E is represented by T and S. We note that $\{e_n + e_{n+1}\}_{n=1}^{\infty}$ is linearly independent.

In general if $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is linearly independent with a Fibonacci representation T, then for each $g \in \operatorname{span}\{f_n\}_{n=1}^{\infty}$ the linear operator $S : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ defined by

$$S\Big(\sum_{i=1}^{k} c_i f_i\Big) = \sum_{i=1}^{k} c_i T f_i + \sum_{i=1}^{k}(-1)^i c_i g$$

is a Fibonacci representation for $\{f_n\}_{n=1}^{\infty}$.

Now, we want to get a sufficient condition for a frame $F = \{f_n\}_{n=1}^{\infty}$ to have a Fibonacci representation. We need the following lemma.

Lemma 3.1. *Consider a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. Then the following hold:*

(*i*) *For $n \geqslant 2$, we have*

$$f_n = \sum_{i=0}^{m-1}(-1)^i(f_{n-i-1} + f_{n-i}) + (-1)^m f_{n-m}, \quad 1 \leqslant m \leqslant n-1.$$

(*ii*) $\operatorname{span}\{f_n\}_{n=1}^{\infty} = \operatorname{span}\big\{\{f_1\} \cup \{f_n + f_{n+1}\}_{n=1}^{\infty}\big\}$.

(*iii*) *If $\{f_n\}_{n=1}^{\infty}$ is linearly independent, then $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ is linearly independent.*

(*iv*) *If $\{f_1\} \cup \{f_n + f_{n+1}\}_{n=1}^{\infty}$ is linearly independent, then $\{f_n\}_{n=1}^{\infty}$ is linearly independent.*

Proof. (i) Let $n \geqslant 2$ and $1 \leqslant m \leqslant n-1$. Then we have

$$\sum_{i=0}^{m-1}(-1)^i(f_{n-i-1}+f_{n-i}) = \sum_{i=1}^{m-1}(-1)^{i-1}f_{n-i} + (-1)^{m-1}f_{n-m} + f_n + \sum_{i=1}^{m-1}(-1)^i f_{n-i}$$
$$= (-1)^{m-1}f_{n-m} + f_n.$$

For the proof of (ii), it is clear that $\operatorname{span}\{\{f_n+f_{n+1}\}_{n=1}^{\infty} \cup \{f_1\}\} \subseteq \operatorname{span}\{f_n\}_{n=1}^{\infty}$. On the other hand, by (i) (for $m = n-1$) we infer $\operatorname{span}\{f_n\}_{n=1}^{\infty} \subseteq \operatorname{span}\{\{f_n+f_{n+1}\}_{n=1}^{\infty} \cup \{f_1\}\}$. This proves (ii). To prove (iii), let $\{c_n\}_{n=1}^{k} \subseteq \mathbb{C}$ such that $\sum_{n=1}^{k} c_n(f_n+f_{n+1}) = 0$. Then we have $c_1f_1 + \sum_{n=2}^{k}(c_{n-1}+c_n)f_n + c_kf_{k+1} = 0$. Since $\{f_n\}_{n=1}^{\infty}$ is linearly independent, we get $c_1 = c_k = 0$ and $c_{n-1}+c_n = 0$ for all $2 \leqslant n \leqslant k$. Therefore, $c_n = 0$ for all $1 \leqslant n \leqslant k$. This completes the proof of (iii).

To prove (iv), let $\{c_n\}_{n=1}^{N} \subseteq \mathbb{C}$ such that $\sum_{n=1}^{N} c_nf_n = 0$. Then by (i), we have

$$0 = \sum_{n=1}^{N} c_nf_n = c_1f_1 + \sum_{n=2}^{N} c_n\Big(\sum_{i=0}^{n-2}(-1)^i(f_{n-i-1}+f_{n-i}) + (-1)^{n-1}f_1\Big)$$
$$= \Big(c_1 + \sum_{n=2}^{N} c_n(-1)^{n-1}\Big)f_1 + \sum_{n=2}^{N} c_n \sum_{i=0}^{n-2}(-1)^i(f_{n-i-1}+f_{n-i})$$
$$= \Big(c_1 + \sum_{n=2}^{N} c_n(-1)^{n-1}\Big)f_1 + \sum_{i=0}^{N-2}\sum_{n=i+2}^{N} c_n(-1)^i(f_{n-i-1}+f_{n-i})$$
$$= \Big(c_1 + \sum_{n=2}^{N} c_n(-1)^{n-1}\Big)f_1 + \sum_{i=2}^{N}\sum_{n=0}^{N-i} c_{i+n}(-1)^i(f_{n+1}+f_{n+2})$$
$$= \Big(c_1 + \sum_{n=2}^{N} c_n(-1)^{n-1}\Big)f_1 + \sum_{n=0}^{N-2}\Big(\sum_{i=2}^{N-n} c_{i+n}(-1)^i\Big)(f_{n+1}+f_{n+2}).$$

Since $\{f_1\} \cup \{f_n+f_{n+1}\}_{n=1}^{\infty}$ is linearly independent, we get

$$c_1 + \sum_{k=2}^{N} c_k(-1)^{k-1} = 0, \qquad \sum_{i=2}^{N-n} c_{i+n}(-1)^i = 0, \quad 0 \leqslant n \leqslant N-2.$$

Hence we conclude that $c_n = 0$ for all $n = 1, 2, .., N$. Then $\{f_n\}_{n=1}^{\infty}$ is linearly independent. □

In the following, we give a sufficient condition for a sequence $F = \{f_n\}_{n=1}^{\infty}$ to have a Fibonacci representation.

Theorem 3.1. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. If $\{f_n+f_{n+1}\}_{n=1}^{\infty}$ is linearly independent, then F has a Fibonacci representation.*

Proof. First we assume that $f_1 \in \operatorname{span}\{f_n+f_{n+1}\}_{n=1}^{\infty}$. Then by (ii) of Lemma 3.1, we have $\operatorname{span}\{f_n+f_{n+1}\}_{n=1}^{\infty} = \operatorname{span}\{f_n\}_{n=1}^{\infty}$. We define a linear operator $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ by

$$T(f_n+f_{n+1}) = f_{n+2}; \quad n \geqslant 2. \tag{2}$$

Since $\{f_n+f_{n+1}\}_{n=1}^{\infty}$ is linearly independent sequence, T is well-defined and F is represented by T. If $f_1 \notin \operatorname{span}\{f_n+f_{n+1}\}_{n=1}^{\infty}$, then $\{f_1\} \cup \{f_n+f_{n+1}\}_{n=1}^{\infty}$ is linearly independent and so by Lemma 3.1 (iv), $\{f_n\}_{n=1}^{\infty}$ is linearly independent. Hence we can define a linear operator $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ by $Tf_n = \sum_{i=0}^{n}(-1)^i f_{n+1-i}$, $n \in \mathbb{N}$. We show that F is represented by T. Indeed,

$$Tf_n + Tf_{n+1} = \sum_{i=0}^{n}(-1)^i f_{n+1-i} + \sum_{i=0}^{n}(-1)^{i+1} f_{n+1-i} + f_{n+2} = f_{n+2}.$$

□

The following example shows that the converse of Theorem 3.1 is not satisfied in general.

Example 3.5. *The frame* $F = \{f_n\}_{n=1}^{\infty} = \{e_1, e_2, e_3, e_2, e_2, e_4, e_5, e_6, \ldots\}$ *is represented by the linear operator* $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ *given by*

$$Te_1 = e_3 - \frac{e_4}{2}, \quad Te_2 = \frac{e_4}{2}, \quad Te_3 = e_2 - \frac{e_4}{2},$$
$$Te_n = \sum_{i=0}^{n-4}(-1)^i e_{n-i+1} + (-1)^{n-3}\frac{e_4}{2}, \quad n \geqslant 4.$$

But $\{f_n + f_{n+1}\}_{n=1}^{\infty} = \{e_1 + e_2, e_2 + e_3, e_3 + e_2, 2e_2, \ldots\}$ *is not linearly independent.*

Corollary 3.1. *Let* $F = \{f_n\}_{n=1}^{\infty}$ *be a linear independent sequence in* $\mathcal{H}$. *Then* F *has a Fibonachi representation.*

Proof. It follows from Lemma 3.1 (iii) and Theorem 3.1. □

Now, we provide sufficient conditions to make the converse of Theorem 3.1 become true.

Theorem 3.2. *Let* $F = \{f_n\}_{n=1}^{\infty}$ *be a complete sequence in an infinite dimensional Hilbert space* $\mathcal{H}$ *which has the Fibonacci representation operator* T. *If there exists* $m \in \mathbb{N}$ *such that* $f_{m+1}, Tf_1 \in \operatorname{span}\{f_n\}_{n=1}^{m}$, *then* $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ *is linearly independent.*

Proof. Suppose that $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ is not linearly independent. Then there exists $n_0 \in \mathbb{N}$ such that $f_{n_0} + f_{n_0+1} = \sum_{n=1}^{n_0-1} c_n(f_n + f_{n+1})$. Hence

$$f_{n_0+2} = T(f_{n_0} + f_{n_0+1}) = \sum_{n=1}^{n_0-1} c_n f_{n+2} \in \operatorname{span}\{f_n\}_{n=1}^{n_0+1}. \tag{3}$$

Let $V = \operatorname{span}\{f_n\}_{n=1}^{l}$, where $l = \max\{n_0+1, m\}$. By (3) and $f_{m+1} \in \operatorname{span}\{f_n\}_{n=1}^{m}$, we get $f_{l+1} \in V$. We show V is invariant under T. Suppose that $f = \sum_{n=1}^{l} c_n f_n \in V$. By using (i) of Lemma 3.1, we have

$$Tf = c_1 Tf_1 + \sum_{n=2}^{l} c_n T\Big(\sum_{i=0}^{n}(-1)^i(f_{n-i-1} + f_{n-i}) + (-1)^{n-1}f_1\Big)$$
$$= \Big(c_1 + \sum_{n=2}^{l} c_n(-1)^{n-1}\Big)Tf_1 + \sum_{n=2}^{l} c_n \sum_{i=0}^{n}(-1)^i f_{n-i+1}.$$

Since $Tf_1 \in \operatorname{span}\{f_n\}_{n=1}^{m} \subseteq V$ and $f_{l+1} \in V$, the above argument proves that V is invariant under T. Therefore $f_n \in V$ for all $n \geqslant l+1$ and consequently $\operatorname{span}\{f_n\}_{n=1}^{\infty} = V$. Since

$\{f_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$, we have $\mathcal{H} = \overline{\operatorname{span}}\{f_n\}_{n=1}^{\infty} = \overline{V} = V$ which is in contradiction to $\dim \mathcal{H} = \infty$. □

Proposition 3.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a complete and linearly dependent sequence with $f_1 \neq 0$ in an infinite dimensional Hilbert space $\mathcal{H}$. Then there exists $m \geqslant 2$ such that $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m}$.*

Proof. Since $\{f_n\}_{n=1}^{\infty}$ is linearly dependent, there exists $k \geqslant 2$ such that $f_k \in \operatorname{span}\{f_n\}_{n=1}^{k-1}$. We claim that there exists an integer $l > k$ such that $f_l \notin \operatorname{span}\{f_n\}_{n=1}^{l-1}$. If $f_l \in \operatorname{span}\{f_n\}_{n=1}^{l-1}$ for each $l > k$, then $f_{k+1} \in \operatorname{span}\{f_n\}_{n=1}^{k-1}$ because $f_k \in \operatorname{span}\{f_n\}_{n=1}^{k-1}$ and $f_{k+1} \in \operatorname{span}\{f_n\}_{n=1}^{k}$. Hence by induction we get $f_l \in \operatorname{span}\{f_n\}_{n=1}^{k-1}$ for each $l > k$. Therefore $\operatorname{span}\{f_n\}_{n=1}^{\infty} = \operatorname{span}\{f_n\}_{n=1}^{k-1}$. Since $\{f_n\}_{n=1}^{\infty}$ is complete and $\dim \mathcal{H} = \infty$, the contradiction is achieved. Now, let $i \in \mathbb{N}$ be the smallest number such that $f_{k+i} \notin \operatorname{span}\{f_n\}_{n=1}^{k+i-1}$. Putting $m = k+i-1$, we get $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m}$. □

Proposition 3.3. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ which is represented by T. Suppose that $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m-1}$ for some integer $m \geqslant 2$. Then $Tf_i \in \operatorname{span}\{f_n\}_{n=3}^{m+1}$ for $1 \leqslant i \leqslant m$.*

Proof. By the assumption, we have $f_m = \sum_{n=1}^{m-1} c_n f_n$, so

$$\begin{aligned}
&\sum_{n=1}^{m-2}\Big(\sum_{i=0}^{n-1}(-1)^i c_{n-i}\Big)(f_n + f_{n+1})\\
&= \sum_{n=1}^{m-2}\Big(\sum_{i=0}^{n-1}(-1)^i c_{n-i}\Big)f_n + \sum_{n=2}^{m-1}\Big(\sum_{i=0}^{n-2}(-1)^i c_{n-i-1}\Big)f_n\\
&= c_1 f_1 + \sum_{n=2}^{m-2}\Big(c_n + \sum_{i=1}^{n-1}(-1)^i c_{n-i} + \sum_{i=1}^{n-1}(-1)^{i-1} c_{n-i}\Big)f_n + \Big(\sum_{i=0}^{m-3}(-1)^i c_{m-i-2}\Big)f_{m-1}\\
&= c_1 f_1 + \sum_{n=2}^{m-2} c_n f_n + \Big(\sum_{i=0}^{m-3}(-1)^i c_{m-i-2}\Big)f_{m-1}\\
&= f_m + \Big(-c_{m-1} + \sum_{i=0}^{m-3}(-1)^i c_{m-i-2}\Big)f_{m-1} = f_m + \Big(\sum_{i=0}^{m-2}(-1)^{i-1} c_{m-i-1}\Big)f_{m-1},
\end{aligned}$$

thus

$$f_{m-1} + f_m = \sum_{n=1}^{m-2}\Big(\sum_{i=0}^{n-1}(-1)^i c_{n-i}\Big)(f_n + f_{n+1}) + \Big(1 - \sum_{i=0}^{m-2}(-1)^{i-1} c_{m-i-1}\Big)f_{m-1}. \tag{4}$$

Since F is represented by T, the equality (4) implies that

$$f_{m+1} = \sum_{n=1}^{m-2}\Big(\sum_{i=0}^{n-1}(-1)^i c_{n-i}\Big)f_{n+2} + \Big(1 - \sum_{i=0}^{m-2}(-1)^{i-1} c_{m-i-1}\Big)Tf_{m-1}. \tag{5}$$

If $1 - \sum_{i=0}^{m-2}(-1)^{i-1}c_{m-i-1} = 0$, then $f_{m+1} \in \operatorname{span}\{f_n\}_{n=3}^{m} \subseteq \operatorname{span}\{f_n\}_{n=1}^{m-1}$ which is a contradiction. Hence (5) implies that

$$Tf_{m-1} = \frac{f_{m+1} - \sum_{n=1}^{m-2}\Big(\sum_{i=0}^{n-1}(-1)^i c_{n-i}\Big)f_{n+2}}{1 - \sum_{i=0}^{m-2}(-1)^{i-1} c_{m-i-1}} \in \operatorname{span}\{f_n\}_{n=3}^{m+1}. \tag{6}$$

Also, by (i) of Lemma 3.1, for $1 \leqslant j \leqslant m-1$, we have

$$f_{m+1} = Tf_m + Tf_{m-1} = \sum_{i=0}^{j-1} (-1)^i f_{m-i+1} + (-1)^j Tf_{m-j} + Tf_{m-1}.$$

Therefore

$$Tf_{m-j} = (-1)^j \big(f_{m+1} - Tf_{m-1} - \sum_{i=0}^{j-1} (-1)^i f_{m-i+1} \big). \tag{7}$$

Hence it follows from (6) and (7) that $Tf_i \in \operatorname{span}\{f_n\}_{n=3}^{m+1}$ for each $1 \leqslant i \leqslant m-1$. □

Corollary 3.2. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ which is represented by T. Suppose that $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m-1}$ for some $m \in \mathbb{N}$. Then, $Tf_{m+i} \in \operatorname{span}\{f_n\}_{n=3}^{m+i+1}$ for each $i \in \mathbb{N}$.*

Proof. Since $Tf_{m+i} = f_{m+i+1} - Tf_{m+i-1}$, the result follows by induction on i and Proposition 3.3. □

Corollary 3.3. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a complete sequence in an infinite dimensional Hilbert space $\mathcal{H}$.*

- (i) *If F is linearly independent, then it has a Fibonacci representation T such that $\mathcal{R}(T) = \operatorname{span}\{f_n\}_{n=3}^{\infty}$.*
- (ii) *If F is linearly dependent, then for every Fibonacci representation T of F we have $\mathcal{R}(T) = \operatorname{span}\{f_n\}_{n=3}^{\infty}$.*

Proof. First we note that if F is represented by T, then $f_n = T(f_{n-1} + f_{n-2}) \in \mathcal{R}(T)$ for every $n \geqslant 3$, and consequently $\operatorname{span}\{f_n\}_{n=3}^{\infty} \subseteq \mathcal{R}(T)$.

To prove (i), consider the linear operator $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ defined by

$$Tf_1 = Tf_2 = \frac{1}{2} f_3, \quad Tf_n = \sum_{i=0}^{n-3} (-1)^i f_{n+1-i} + \frac{(-1)^n}{2} f_3, \quad n \geqslant 3.$$

Then $Tf_1 + Tf_2 = f_3$, $Tf_2 + Tf_3 = f_4$ and

$$Tf_n + Tf_{n+1} = \sum_{i=0}^{n-3} (-1)^i f_{n-i+1} + \sum_{i=0}^{n-2} (-1)^i f_{n-i+2} = f_{n+2}, \quad n \geqslant 3.$$

Hence F is represented by T and it is obvious that $\mathcal{R}(T) \subseteq \operatorname{span}\{f_n\}_{n=3}^{\infty}$. In order to prove (ii), by Proposition 3.2 there exists $m \geqslant 2$ such that $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m-1}$. If F is represented by T, then by Proposition 3.3 and Corollary 3.2 we have $\mathcal{R}(T) \subseteq \operatorname{span}\{f_n\}_{n=3}^{\infty}$. □

In Theorem 3.2, we showed that $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ is linearly independent under some conditions. In the following, we show that (under some conditions) by removing finitely many elements of $\{f_n + f_{n+1}\}_{n=1}^{\infty}$ the remaining elements will be linearly independent.

Theorem 3.3. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a complete sequence in an infinite dimensional Hilbert space $\mathcal{H}$ which is represented by T. Then there exists $m \in \mathbb{N}$ such that $\{f_{m+n} + f_{m+n+1}\}_{n=1}^{\infty}$ is linearly independent.*

Proof. If F is linearly independent, then the result follows by (iii) of Lemma 3.1. Suppose that F is linearly dependent. Then by Proposition 3.2 and Proposition 3.3, there exists $m \geqslant 2$ such that $f_m \in \operatorname{span}\{f_n\}_{n=1}^{m-1}$, $f_{m+1} \notin \operatorname{span}\{f_n\}_{n=1}^{m-1}$ and $Tf_1 \in \operatorname{span}\{f_n\}_{n=3}^{m+1}$. We prove $\{f_{m+n} + f_{m+n+1}\}_{n=1}^{\infty}$ is linearly independent. Suppose by contradiction that $\{f_{m+n}+f_{m+n+1}\}_{n=1}^{\infty}$ is not linearly independent. Then there exists $j \in \mathbb{N}$ such that $f_{m+j} + f_{m+j+1} = \sum_{n=1}^{j-1} c_n(f_{m+n} + f_{m+n+1})$. Hence we have

$$f_{m+j+2} = T(f_{m+j} + f_{m+j+1}) = \sum_{n=1}^{j-1} c_n f_{m+n+2} \in \operatorname{span}\{f_n\}_{n=1}^{m+j+1}. \tag{8}$$

Let $V = \operatorname{span}\{f_n\}_{n=1}^{m+j+1}$. We show that V is invariant under T. Let $f = \sum_{n=1}^{m+j+1} c_n f_n \in V$. Then by (i) of Lemma 3.1, we have

$$Tf = \Big(c_1 + \sum_{n=2}^{m+j+1} c_n(-1)^{n-1}\Big)Tf_1 + \sum_{n=2}^{j+m+1} c_n \sum_{i=0}^{n-2}(-1)^i f_{n-i+1}.$$

Using $Tf_1 \in \operatorname{span}\{f_n\}_{n=3}^{m+1} \subseteq V$ and (8), we get $Tf \in V$. Then we conclude $f_n \in V$ for all $n \geqslant m + j + 2$. Thus, $\operatorname{span}\{f_n\}_{n=1}^{\infty} = V$ and since $\{f_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$, we have $\mathcal{H} = \overline{\operatorname{span}}\{f_n\}_{n=1}^{\infty} = \overline{V} = V$ which is a contradiction. □

4. **Fibonacci Representation Operators**

In a frame that indeed has the form $\{T^n\varphi\}_{n=0}^{\infty}$, where $T \in B(\mathcal{H})$ and $\varphi \in \mathcal{H}$, all sequence members are represented by iterative actions of T on φ. In the case where $\{f_n\}_{n=1}^{\infty}$ has a Fibonacci representation operator T, we expect (Theorem 4.1) all members of the sequence $\{f_n\}_{n=1}^{\infty}$ to be identified in terms of iterative actions of T on elements f_1 and f_2. In this section, we present some results concerning Fibonacci representation operators. One of the results characterizes types of frame which can be represented in terms of a bounded operator T.

Notation. $[x]$ denotes the integer part of $x \in \mathbb{R}$ and $\begin{pmatrix} n \\ k \end{pmatrix} := \frac{n!}{k!(n-k)!}$ for integers $0 \leqslant k \leqslant n$. We let $\begin{pmatrix} n \\ k \end{pmatrix} := 0$ when $k > n$ or $k < 0$.

Theorem 4.1. *Let $T : \operatorname{span}\{f_n\}_{n=1}^{\infty} \to \operatorname{span}\{f_n\}_{n=1}^{\infty}$ be a linear operator, then the following statements are equivalent:*

(i) *$F = \{f_n\}_{n=1}^{\infty}$ is represented by T.*

(ii) *$Tf_1 + Tf_2 = f_3$ and for $a_n = [\frac{n-1}{2}]$, $b_n = n - 2a_n - 2$,*

$$f_n = \sum_{i=a_n}^{2a_n} \Big(\begin{pmatrix} i + b_n \\ 2i - 2a_n + b_n \end{pmatrix} T^{i+b_n} f_2 + \begin{pmatrix} i + b_n \\ 2i - 2a_n + b_n + 1 \end{pmatrix} T^{i+b_n+1} f_1 \Big), \; n \geqslant 4. \tag{9}$$

Proof. $(i) \Rightarrow (ii)$ We prove (9) by induction on n. For $n = 4$, we have $a_4 = 1$ and $b_4 = 0$. Then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} Tf_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} T^2 f_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} T^2 f_2 + \begin{pmatrix} 2 \\ 3 \end{pmatrix} T^3 f_1 = f_4.$$

Now, assume that $k > 4$ and (9) holds for all $n \leqslant k$ and we prove (9) for $n = k+1$. If $k+1$ is even, then $b_k = -1,\ b_{k-1} = b_{k+1} = 0$ and $a_{k+1} = a_k = 1 + a_{k-1}$. Hence

$$\begin{aligned} f_{k+1} = Tf_k + Tf_{k-1} &= \sum_{i=a_k}^{2a_k} \Big(\binom{i-1}{2i-2a_k-1} T^i f_2 + \binom{i-1}{2i-2a_k} T^{i+1} f_1 \Big) \\ &+ \sum_{i=a_k}^{2a_k} \Big(\binom{i-1}{2i-2a_k} T^i f_2 + \binom{i-1}{2i-2a_k+1} T^{i+1} f_1 \Big) \\ &= \sum_{i=a_k}^{2a_k} \Big(\binom{i}{2i-2a_k} T^i f_2 + \binom{i}{2i-2a_k+1} T^{i+1} f_1 \Big). \end{aligned}$$

Since $b_{k+1} = 0$ and $a_{k+1} = a_k$, we obtain (9). If $k+1$ is odd, then $b_k = 0,\ b_{k-1} = b_{k+1} = -1$ and $1 + a_{k-1} = 1 + a_k = a_{k+1}$. Hence

$$f_{k+1} = Tf_k + Tf_{k-1} = \sum_{i=a_{k+1}}^{2a_{k+1}} \Big(\binom{i-1}{2i-2a_{k+1}-1} T^{i-1} f_2 + \binom{i-1}{2i-2a_{k+1}} T^i f_1 \Big).$$

Hence we get (9). To prove $(ii) \Rightarrow (i)$, there are two possibilities. If $n > 4$ is odd, then $b_n = -1,\ b_{n-1} = b_{n+1} = 0$ and $a_{n+1} = a_n = 1 + a_{n-1}$. Hence

$$\begin{aligned} Tf_n &= \sum_{i=a_n}^{2a_n} \Big(\binom{i-1}{2i-2a_n-1} T^i f_2 + \binom{i-1}{2i-2a_n} T^{i+1} f_1 \Big), \\ Tf_{n-1} &= \sum_{i=a_n}^{2a_n} \Big(\binom{i-1}{2i-2a_n} T^i f_2 + \binom{i-1}{2i-2a_n+1} T^{i+1} f_1 \Big). \end{aligned}$$

Therefore,

$$Tf_n + Tf_{n-1} = \sum_{i=a_n}^{2a_n} \Big(\binom{i}{2i-2a_n} T^i f_2 + \binom{i}{2i-2a_n+1} T^{i+1} f_1 \Big) = f_{n+1}.$$

If $n \geqslant 4$ is even, the argument is similar to the previous case. □

Remark 4.1. We recall that

$$\ell^2(\mathcal{H}) := \Big\{ \{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H} : \sum_{n=1}^{\infty} \|f_n\|^2 < \infty \Big\},$$

and $\mathcal{T}_L, \mathcal{T}_R : \ell^2(\mathcal{H}) \to \ell^2(\mathcal{H})$ are bounded linear operators defined by

$$\mathcal{T}_L\{f_n\}_{n=1}^{\infty} = \{f_{n+1}\}_{n=1}^{\infty}, \quad \mathcal{T}_R\{f_n\}_{n=1}^{\infty} = \{0, f_1, f_2, f_3, ...\}.$$

Proposition 4.1. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ which is represented by T_0 and let $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. Then $\ker T_M \subseteq \ker T_{\mathcal{T}_L^2 F}$ if and only if $T := T_0|_{\operatorname{span}\{f_n+f_{n+1}\}_{n=1}^{\infty}}$ is bounded with $\|T\| \leqslant \sqrt{\frac{B_F}{A_M}}$, where B_F is a Bessel bound for F and A_M is a lower frame bound for M.*

Proof. Let $\ker T_M \subseteq \ker T_{\mathcal{T}_L^2 F}$ and $f = \sum_{n=1}^{k} c_n (f_n + f_{n+1})$, where $\{c_n\}_{n=1}^{\infty} \in \ell^2$ with $c_n = 0$ for $n \geqslant k+1$. Then

$$Tf = \sum_{n=1}^{k} c_n T(f_n + f_{n+1}) = \sum_{n=1}^{\infty} c_n f_{n+2} = \sum_{n=1}^{\infty} d_n f_{n+2} + \sum_{n=1}^{\infty} r_n f_{n+2},$$

where $\{d_n\}_{n=1}^{\infty} \in \ker T_M \subseteq \ker T_{\mathcal{T}_L^2 F}$ and $\{r_n\}_{n=1}^{\infty} \in (\ker T_M)^{\perp}$. Since $\{d_n\}_{n=1}^{\infty} \in \ker T_{\mathcal{T}_L^2 F}$, we have $\sum_{n=1}^{\infty} d_n f_{n+2} = 0$ and consequently $Tf = \sum_{n=1}^{\infty} r_n f_{n+2}$. Therefore by applying [4, Lemma 5.5.5], we have

$$\|Tf\|^2 \leqslant \frac{B_F}{A_M}\Big\|\sum_{n=1}^{\infty} r_n(f_n + f_{n+1})\Big\|^2 = \frac{B_F}{A_M}\Big\|\sum_{n=1}^{k} c_n(f_n + f_{n+1})\Big\|^2 = \frac{B_F}{A_M}\|f\|^2.$$

For the other implication, let $\{c_n\}_{n=1}^{\infty} \in \ker T_M$. Since $\sum_{n=1}^{\infty} c_n(f_n + f_{n+1}) = 0$ and T is bounded, we have $\sum_{n=1}^{\infty} c_n f_{n+2} = 0$, that means $\{c_n\}_{n=1}^{\infty} \in \ker T_{\mathcal{T}_L^2 F}$. □

Remark 4.2. In Theorem 1.2, the invariance of $\ker T_F$ under the right-shift operator $\mathcal{T}$ is a sufficient condition for the boundedness of T. It is obvious that the invariance of $\ker T_F$ under $\mathcal{T}$ is equivalent to $\ker T_F \subseteq \ker T_{\mathcal{T}_L F}$. In fact, for $\{c_n\}_{n=1}^{\infty} \in \ell^2$ we have $\mathcal{T}(\{c_n\}_{n=1}^{\infty}) \in \ker T_F$ if and only if $\{c_n\}_{n=1}^{\infty} \in \ker T_{\mathcal{T}_L F}$.

Proposition 4.2. *Let $F = \{f_n\}_{n=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ which is represented by $T \in B(\mathcal{H})$ and $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. Then T is injective if and only if $\ker T_{\mathcal{T}_L^2 F} \subseteq \ker T_M$*

Proof. Let T be injective and $\{c_n\}_{n=1}^{\infty} \in \ker T_{\mathcal{T}_L^2 F}$. Then

$$T\Big(\sum_{n=1}^{\infty} c_n(f_n + f_{n+1})\Big) = \sum_{n=1}^{\infty} c_n f_{n+2} = 0.$$

Since T is injective, we get $\sum_{n=1}^{\infty} c_n(f_n + f_{n+1}) = 0$ and consequently $\{c_n\}_{n=1}^{\infty} \in \ker T_M$. Conversely, assume that $f \in \mathcal{H}$ and $Tf = 0$. Since $M = \{f_n + f_{n+1}\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$, we have $f = \sum_{n=1}^{\infty} c_n(f_n + f_{n+1})$ for some $\{c_n\}_{n=1}^{\infty} \in \ell^2$. Then $\sum_{n=1}^{\infty} c_n f_{n+2} = 0$, and so $\{c_n\}_{n=1}^{\infty} \in \ker T_{\mathcal{T}_L^2 F} \subseteq \ker T_M$. This means $f = \sum_{n=1}^{\infty} c_n(f_n + f_{n+1}) = 0$ and the proof is completed. □

Proposition 4.3. *Let $\{f_n\}_{n=1}^{\infty}$ be represented by T. Then the following hold:*

- (*i*) *If $K \in B(\mathcal{H})$ is injective and has closed range, then $\{Kf_n\}_{n=1}^{\infty}$ has a Fibonacci representation.*
- (*ii*) *If $K \in B(\mathcal{H})$ is surjective, then $\{K^* f_n\}_{n=1}^{\infty}$ and $\{KK^* f_n\}_{n=1}^{\infty}$ have Fibonacci representations.*

Proof. (*i*) By Open Mapping Theorem, there exists a bounded linear operator $S : \mathcal{R}(K) \to \mathcal{H}$ such that $SK = I_{\mathcal{H}}$. Therefore

$$KTS(Kf_n + Kf_{n-1}) = KT(f_n + f_{n-1}) = Kf_{n+1}, \quad n \geqslant 2.$$

To prove (*ii*), by [4, Lemma 2.4.1], K^* is injective and has closed range. Also KK^* is invertible. Then (*i*) implies (*ii*). □

Proposition 4.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$ and represented by $T \in B(\mathcal{H})$. If $Tf_1 \in \operatorname{span}\{f_n\}_{n=3}^{\infty}$, then $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \overline{\operatorname{span}}\{Tf_n\}_{n=1}^{\infty} = \overline{\operatorname{span}}\{f_n\}_{n=3}^{\infty}$.*

Proof. For each $f \in \mathcal{H}$, there exists $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that $f = \sum_{n=1}^{\infty} c_n f_n$. Then $Tf = \sum_{n=1}^{\infty} c_n Tf_n \in \overline{\operatorname{span}}\{Tf_n\}_{n=1}^{\infty}$, and therefore $\mathcal{R}(T) \subseteq \overline{\operatorname{span}}\{Tf_n\}_{n=1}^{\infty}$. On the other hand, for $g \in \overline{\text{span}}\{f_n\}_{n=3}^{\infty}$ there exists $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that

$$g = \sum_{n=1}^{\infty} c_n f_{n+2} = \sum_{n=1}^{\infty} c_n T(f_n + f_{n+1}) = T\Big(\sum_{n=1}^{\infty} c_n (f_n + f_{n+1})\Big) \in \mathcal{R}(T).$$

Then $\overline{\text{span}}\{f_n\}_{n=3}^{\infty} \subseteq \mathcal{R}(T)$. Since $Tf_1 \in \text{span}\{f_n\}_{n=3}^{\infty}$, we can now apply (ii) of Lemma 3.1 to conclude that

$$\text{span}\{Tf_n\}_{n=1}^{\infty} = \text{span}\Big\{\{Tf_1\} \cup \{Tf_n + Tf_{n+1}\}_{n=1}^{\infty}\Big\} = \text{span}\{f_n\}_{n=3}^{\infty}.$$

Therefore $\mathcal{R}(T) = \overline{\text{span}}\{Tf_n\}_{n=1}^{\infty} = \overline{\text{span}}\{f_n\}_{n=3}^{\infty}$. $\square$

Proposition 4.5. *Let $\{f_n\}_{n=1}^{\infty}$ be a linearly dependent frame sequence represented by $T \in B(\mathcal{K})$, where $\mathcal{K} = \overline{\text{span}}\{f_n\}_{n=1}^{\infty}$ is an infinite dimensional Hilbert space. Then $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \overline{\text{span}}\{f_n\}_{n=3}^{\infty}$.*

Proof. Let T_0 be the restriction of T on $\text{span}\{f_n\}_{n=1}^{\infty}$. Then by (ii) of Corollary 3.3, we have $\mathcal{R}(T_0) = \text{span}\{f_n\}_{n=3}^{\infty}$, and therefore $\mathcal{R}(T) \subseteq \overline{\text{span}}\{f_n\}_{n=3}^{\infty}$. On the other hand, Since $\{f_n\}_{n=1}^{\infty}$ is a frame sequence, for each $f \in \overline{\text{span}}\{f_n\}_{n=3}^{\infty}$, there exists $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that

$$f = \sum_{n=1}^{\infty} c_n f_{n+2} = \sum_{n=1}^{\infty} c_n (Tf_n + Tf_{n+1}) = T\Big(\sum_{n=1}^{\infty} c_n (f_n + f_{n+1})\Big) \in \mathcal{R}(T).$$

Hence $\overline{\text{span}}\{f_n\}_{n=3}^{\infty} \subseteq \mathcal{R}(T)$ and this completes the proof. $\square$

Theorem 4.2. *Let $\{f_n\}_{n=1}^{\infty}$ be represented by T and S. If $Tf_1 = Sf_1$, then $T = S$ on* $\text{span}\{f_n\}_{n=1}^{\infty}$.

Proof. Since $Tf_1 = Sf_1$ and $T(f_n + f_{n+1}) = f_{n+2} = S(f_n + f_{n+1})$ for all $n \in \mathbb{N}$, we get $Tf_n = Sf_n$ for all $n \in \mathbb{N}$ (we can use (i) of Lemma 3.1). This proves $T = S$ on $\text{span}\{f_n\}_{n=1}^{\infty}$. $\square$

Corollary 4.1. *Let $\{f_n\}_{n=1}^{\infty}$ be represented by T and S. If $f_1 \in \text{span}\{f_n + f_{n+1}\}_{n=k}^{\infty}$ for some $k \in \mathbb{N}$, then $T = S$.*

Proof. Since $f_1 \in \text{span}\{f_n + f_{n+1}\}_{n=k}^{\infty}$, we have $f_1 = \sum_{n=k}^{m} c_n (f_n + f_{n+1})$ for some scalars $c_k, \cdots, c_m$. Then

$$Tf_1 = \sum_{n=k}^{m} c_n T(f_n + f_{n+1}) = \sum_{n=k}^{m} c_n f_{n+2} = \sum_{n=k}^{m} c_n S(f_n + f_{n+1}) = Sf_1.$$

Therefore $T = S$ by Theorem 4.2. $\square$

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